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High School Math Badging System

Implementation Guide

Developed for XQ Institute
By Student Achievement Partners

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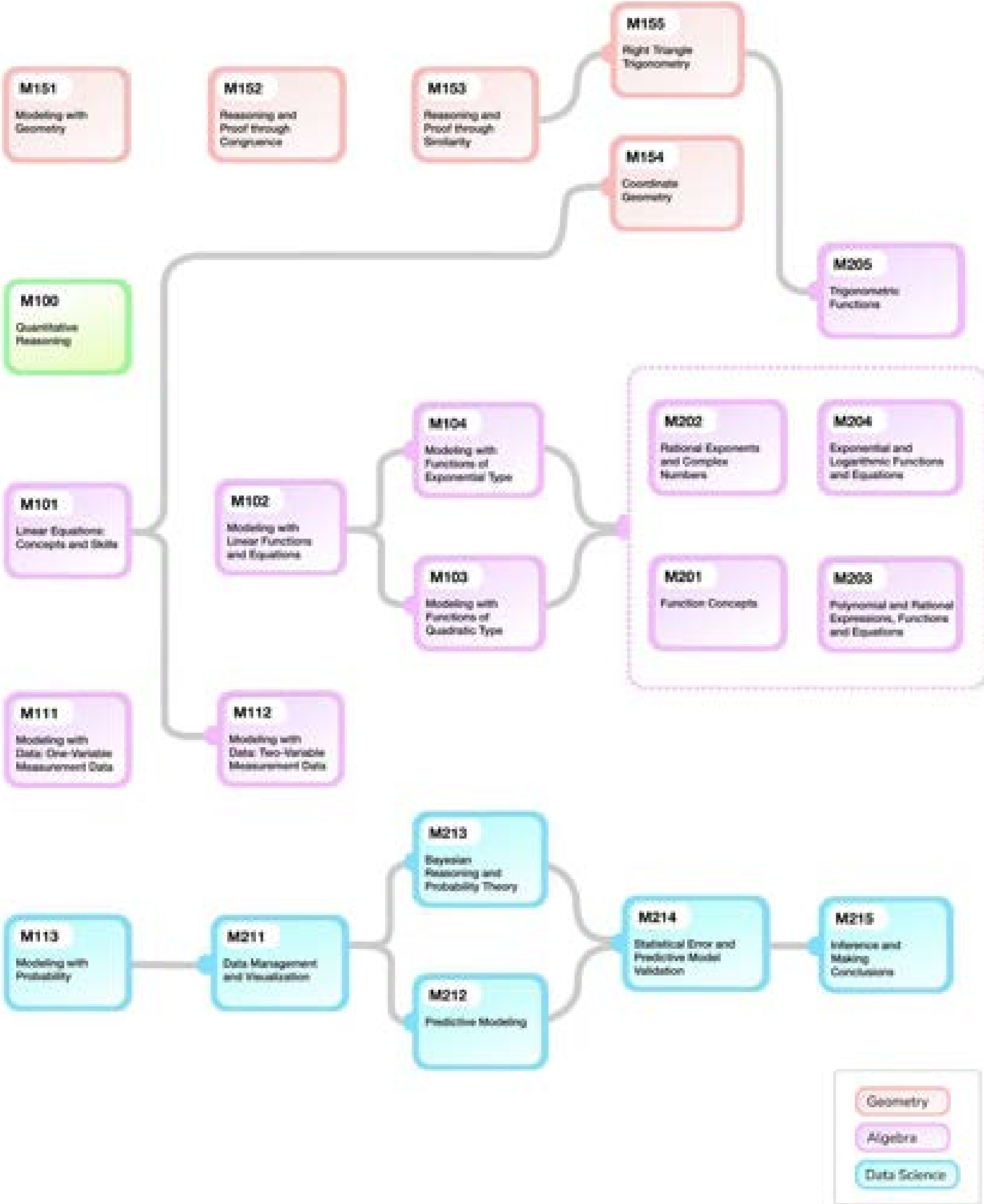
Introduction

The XQ Institute and Student Achievement Partners have developed a vision for a high school math badging system to energize high school mathematics. High school mathematics must prepare students for a diverse set of outcomes: college readiness, STEM-major college readiness, and career readiness. Striking this balance is difficult. Meanwhile, math course-taking decisions and achievement levels in high school limit many students' ability to pursue STEM majors and careers altogether.

A high school math badging system would offer an alternative credentialing mechanism, wherein students would be able to certify learning gained through a broad range of sources (coursework, independent study, summer school, etc.). For example, a student may learn the content for a data analysis badge in several weeks' time through independent study and earn a badge successfully, regardless of their performance in a related course. This badge would then be recognized by high schools, colleges, and employers as a valuable subset of prerequisite concepts and skills.

High School Math Badging Framework



High School Math Badge Frameworks for Curriculum and Assessment Development

Badge frameworks are designed to be used by content developers to create curriculum and assessment materials. The following principles describe how mathematics content is selected and presented in the badge framework. They are united in a vision of student agency as central to student engagement in classroom content. Teachers may also use these frameworks to align existing instructional resources with the badges' content and add supplemental resources to complete the badge requirements.

Framework Design Principles

1. **Equity.** Throughout history, Black, Indigenous, and other students of color have been systematically denied opportunities to learn mathematics, and have instead been stigmatized and tracked into lower-level classes, then labeled as unworthy and inferior. These labels, rooted in stereotypes of Black and brown children, carry over into classrooms and other parts of the educational system, robbing students of opportunities to showcase their brilliance. These badge frameworks describe mathematics instructional materials and assessments anchored in rich and culturally relevant learning experiences designed to disrupt these unjust patterns of marginalization in mathematics education.
2. **Relevance and Connections.** These badge frameworks are designed to focus on content with practical uses. While some topics may be selected for their relevance to college-level coursework, others are chosen for their relevance to career pathways. The combination of experiential and academic learning will provide students with direct connections between the classroom and real life. The frameworks are designed to emphasize real-world connections and problem-solving experiences that bring the learning of mathematics to life in ways that directly connect with students' lived experiences.
3. **Depth and Coherence.** Too often, students are asked to sit through year-long courses that survey disparate ideas without connection, appropriate depth, and relevance. These badge frameworks describe courses that focus on key ideas with adequate depth of understanding and meaningful connections across mathematical ideas, where coherence is established through applicable topics.
4. **Flexibility and Choice.** In order to disrupt inequities, badge coursework must be made available in a variety of settings and formats. As a result, these badge frameworks, while clearly specifying summative outcomes and learning experiences, offer little in the way of traditional course requirements like seat time, credit hours, or numbers of instructional days. The frameworks are thus designed to encourage a varied ecosystem of options (remote, in-person, asynchronous, etc.) for developers to consider and ultimately, for students to choose from.
5. **Action-Oriented.** To easily facilitate content development, these frameworks include many concrete examples and learning experiences. They are designed to paint a crisp picture of student learning that readily informs the creation of instructional materials and assessments.

Badge Framework Components

Each badge framework features three central components:
(1) Mathematical Content and Practice Expectations

(2) Guidance for Designing: Learning Principles
(3) Evidence of Learning

These are described in detail in the sections that follow.

Mathematical Content and Practice Expectations

Each badge framework describes the mathematics skills and understandings that students will need to demonstrate to earn each badge. These expectations should connect to the learning experiences that students engage in and to the evidence of learning that students use to demonstrate their understanding of the expectations.

Guidance for Designing: Learning Principles

Each badge framework employs learning principles students should engage in to support them in meeting the content and practice expectations for the badge. These learning principles should undergird the development of instructional materials for badges. Content development for the different badges should engage students in experiences supported by these Learning Principles (LPs):

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

To illustrate how the Learning Principles intersect with the frameworks, each framework describes the **Points of Emphasis** for the badge and has **Annotated Examples (Optional)** of learning experiences that can be used to support students' engagement with the content and practices. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles less likely to be part of published curricular materials for mathematics instruction.

Evidence of Learning

The goal of the badging system is for each student who attempts a badge to earn it. This is in stark contrast to how the current mathematics education system is set up: mostly as a filtering and sorting mechanism that presents obstacles and provides little opportunity for students to exercise autonomy in the ways that they demonstrate evidence of their learning. It is important to state this upfront because a more humanizing mathematics experience means that we must think differently about how we provide opportunities for students to earn a badge and how we ensure their success along the way.

Each badge framework describes the forms of evidence of student learning that can be used to award the badge credit. These different forms of evidence will be flexibly integrated into the badge frameworks to ensure an appropriate match between the evidence required to earn a badge and the content and practice demands of the badge.

Forms of Evidence

The sections that follow provide information about each of the assessment types listed in the table below. Students earn badges by providing evidence that demonstrates mastery of the badge content and practice expectations in a multiple measures system.

Badge	Portfolio of Evidence	Concepts and Skills Assessment	Performance Assessment
M100	X	X	
M101	X	X	
M102	X	X	X
M103	X	X	X
M104	X	X	X
M111	X	X	X
M112	X	X	X
M113	X	X	X
M151	X	X	X
M152	X	X	
M153	X	X	
M154	X	X	
M155	X	X	
M201	X	X	
M202	X	X	
M203	X	X	
M204	X	X	
M205	X	X	
M211	X	X	
M212	X	X	X
M213	X	X	
M214	X	X	
M215	X	X	

Portfolio of Evidence

The portfolio of evidence is a set of artifacts aligned to the indicators collected throughout the students' interactions with the badge content through their learning experiences. Students are given agency to choose artifacts that make them proud and allow them to demonstrate their learning through multiple modalities. This provides students with a voice and reflects the mindsets and habits of professionals who use mathematics in their work. Students are given the opportunity to showcase evidence of their own learning through the work they have done.

The portfolio is used in conjunction with a concepts and skills assessment for non-modeling badges. These two forms of assessment provide multiple measures of student learning of the badge Content and Practice Expectations. The evidence collected in both assessments validates the demonstration of mastery for earning the badge. For modeling badges, at least one performance assessment will be included in a student's portfolio to demonstrate successful engagement with the full modeling cycle (Sackstein, 2019).

Concepts and Skills Assessment

The concepts and skills assessment is a standardized, computer-based assessment that provides evidence of the range of content and practice expectations for the badge. A blueprint for each badge is provided in Appendix B, laying out the specific content and practice expectations, the cognitive complexity focus, and the number of items needed for each expectation. For modeling badges, the assessment items are tied to common stimuli to better reflect the nature of mathematical modeling. The assessments and the conditions under which the assessments are administered should allow maximum flexibility and support the overarching goal of students earning the badges. The following criteria should be adhered to in assessment design and administration:

Badge assessments are criterion referenced. The purpose of the assessment is to support the determination of whether a student has earned the badge. Its purpose is not to compare students to each other. Discrimination is not information, but comparison to criteria is. The use of the IRT test methodology does not fit the purposes of badge assessment. If 100% of students accomplish what is expected, the assessment should award badges to 100% of students.

Minimize false negatives. Items on a badge exam assessing concepts and skills should minimize construct-irrelevant difficulties. Assessments should include easy items that provide the evidence stipulated in the blueprint. Items should not go beyond the blueprint evidence stipulations by incorporating excessive complexity, miscues, messy numbers, inconsiderate wording, unfamiliar prompting, or other demands that increase the probability of execution errors for a student who understands the concept or holds the skill.

Provide sufficient time. There should be ample time for all students. How fast one works is not being assessed.

Allow students to retake the assessment. There are many reasons why students may not be initially successful on an assessment of this type. Allowing students multiple opportunities to take the assessment, sometimes after additional learning, is an important part of a success-focused assessment system.

Performance Assessment

The performance assessment is described in Appendix C and provides students with the opportunity to show evidence of their learning specific to the particulars of badges that center mathematical modeling and application, allowing them authentic ways to demonstrate their ability to apply transferable and real-world skills. For modeling badges, students will complete the full modeling cycle grounded in engaging and meaningful context(s) and report their conclusions using modes and means that are well suited to the overall purpose of the task and that allow for some agency. The following criteria are offered for designing high-quality performance assessments (Safir & Dugan, 2021):

1. Elicits evidence of skills and knowledge matter.
2. Is tight on quality criteria while open to different approaches.
3. Is authentic.
4. Offers a learning experience in and of itself.

For modeling badges and other badges strongly grounded in the application of mathematics, performance assessment will be a required component of the badge earning process. The framework will provide a list of required content, which should not be treated as a checklist for development, but rather as an integrated set of skills that should be present as part of the task. The primary focus for cognitive complexity for the performance tasks will be Application Level 3, which is described in the following section.

Cognitive Complexity Framework

In addition to attending to the method(s) for eliciting evidence of student learning demanded by the content and topics of the different badges, designers must also reflect the cognitive complexity of badge expectations in the badge earning process. The Evidence of Learning guidance provided within each badge framework will leverage Achieve's tool for evaluating cognitive complexity in mathematics assessments, which builds on earlier work on depth of knowledge, but which provides more specific guidance for varying complexity along the dimensions of procedural skill and fluency, conceptual understanding, and application. Specifically, the badge frameworks will describe only the primary target for cognitive complexity for a task or item, leaving the other sources of complexity more flexible. Table 6 provides an overview of the levels of complexity along these three dimensions (Achieve, 2019).

Table 6. Framework for Evaluating the Cognitive Complexity in Mathematics Assessments

	Level 1	Level 2	Level 3
Procedural Complexity: [13]	Solving the problem entails little procedural[14] demand or procedural demand is below grade level.	Solving the problem entails common or grade-level procedure(s) with friendly numbers.	Solving the problem requires common or grade-level procedure(s) with unfriendly numbers,[15] an unconventional combination of procedures, or requires unusual perseverance or organizational skills in the execution of the procedure(s).
Conceptual Complexity: [16]	Solving the problem requires students to recall or recognize a grade-level concept. The student does not need to relate concepts or demonstrate a line of reasoning.	Students may need to relate multiple grade-level concepts of different types, create multiple representations or solutions, or connect concepts with procedures or strategies. The student must do some reasoning, but may not need to demonstrate a line of reasoning.	Solving the problem requires students to relate multiple grade-level concepts and to evidence reasoning, planning, analysis, judgment, and/or creative thought OR work with a sophisticated (nontypical) line of reasoning.
Application Complexity:	Solving the problem entails an application of mathematics, but the required mathematics is either directly indicated or obvious.	Solving the problem entails an application of mathematics and requires an interpretation of the context to determine the procedure or concept (may include extraneous information). The mathematics is not immediately obvious. Solving the problem requires students to decide what to do.	In addition to an interpretation of the context, solving the problem requires recognizing important features, and formulating, computing, and interpreting results as part of a modeling process.

13. This is based on the NAEP States Item to Item Comparison Study, NAEP Validity Studies Panel; Philip Daro, Gerunda Hughes, Sami Kamito, Fran Stancavage, Natalie Tucker-Bradway; American Institutes for Research, 2018.
14. A procedure is a step by step sequence that can be memorized and executed without understanding or attending to the meaning of the quantities; a procedure is useful for a class of problems or situations. Computations that are likely to be known from memory are considered procedural.
15. Unfriendly numbers: The student is likely to get the problem wrong not because of the targeted procedure but because of the numbers involved.
16. This is based on the conceptual aspects of the Mathematics Framework for the 2015 National Assessment of Educational Progress; National Assessment Governing Board, 2014.

Glossary

Artifact: any piece of student work that demonstrates understanding of a particular indicator. Sometimes, an artifact may illustrate evidence of mastery of more than one indicator.

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High School Math Badge Catalog

M100 Quantitative Reasoning

How long will it take to save up for a new phone? Is affordable housing distributed fairly? Math helps us answer questions like these and reason effectively in a variety of real-world situations, including designing a building, interpreting survey results, or comparing costs of credit card offers. In M100 Quantitative Reasoning, you will make sense of and solve real-world problems by strategically using tools like proportionality and percentages, geometric measurement, and data displays. Using technology as an aid for calculations, you will reason about quantities in different units and use formulas to solve problems. Quantitative reasoning is a critical tool that can lead to more sophisticated modeling with functions and geometry. It is also the bedrock of the mathematics of our daily lives and widely applicable to nearly every professional field.

Suggested prerequisites for this badge: concepts of addition, subtraction, multiplication, and division; fraction and decimal concepts; understanding of positive and negative numbers.

Badges to try next: M111 Modeling with One-Variable Measurement Data; M101 Linear Equations: Concepts and Skills; M102 Modeling with Linear Functions and Equations; M151 Modeling with Geometry.

M101 Linear Equations: Concepts and Skills

How does the graph of a system of linear equations in two variables show the solution to that system? How is the perimeter of a rectangle related to its side lengths? Equations and inequalities are powerful tools allowing us to communicate mathematical relationships, solve problems, and model phenomena. In particular, linear equations and inequalities are useful for finding measurements in a geometric figure, exploring how constraints on production could impact business profits, or analyzing a myriad of other situations. In M101 Linear Equations: Concepts and Skills, you will reason about linear equations and develop fluency with graphical and other methods to determine solutions, using technology as an aid. Understanding linear equations and inequalities helps understand quadratic, exponential, and other equation types, as well as modeling and problem-solving in a variety of increasingly complex contexts. Linear equations and inequalities are useful for careers in a variety of fields, like the sciences, medicine and healthcare, business, and engineering.

Suggested prerequisites for this badge: comfort with solving simple equations and problems involving proportional relationships.

This badge is suggested as a prerequisite for: M112 Modeling with Two-Variable Measurement Data; M154 Coordinate Geometry.

M102 Modeling with Linear Functions and Equations

Which cell phone plan holds the best value for me? What combination of aluminum and iron will make the optimal container? Modeling with functions is a powerful way to gain insight into the world around us. Nature, society, business, and everyday life are full of situations in which a certain quantity depends on other measurable quantities. Function models enable us to describe, analyze, optimize, and predict what could happen in these situations. Meanwhile, equation models allow us to express real-world constraints in mathematical language. Solving the equations gives us new and useful information about the situation. In M102 Modeling with Linear Functions and Equations, you will create function models insituations where changes in one quantity are assumed to be proportional to changes in another quantity. You will solve problems and interpret models to gain insight into situations involving a constant rate, such as the startup cost for a business, constant speed, uniform density, and many others. You will also create linear equations to model constraints in real-world problems, such as problems involving finite resources, spatial constraints, manufacturing specifications, or other conditions that must be satisfied. As you learn linear modeling, you will reason quantitatively using the relationships between the parts of a linear function model and the situation it describes. You will use technology as a tool to solve linear equations and to understand how the graphs of linear functions relate to their constituent parts. The work for this badge builds a foundation for the future study of topics such as quadratic and exponential function models. Modeling with linear functions and equations is useful for careers in a variety of fields, like science, engineering, medicine, business, and artificial intelligence.

Suggested prerequisites for this badge: comfort with solving simple equations and problems involving proportional relationships.

This badge is suggested as a prerequisite for: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type

M103 Modeling with Functions of Quadratic Type

How can we predict the path of a rocket through the sky? What quantity of sales will maximize business profits? Modeling with functions is a powerful way to gain insight into the world around us. Nature, society, business, and everyday life are full of situations in which a certain quantity of interest depends on other measurable quantities. With function models, we can describe and analyze these situations and even optimize them and predict what will happen. In M103 Modeling with Functions of Quadratic Type, you will create and make sense of models in situations that suggest a quadratic relationship. You will solve problems and gain insight into situations like accelerated motion or determining how to produce the best outcome in a model that represents a real-world problem in the workplace, the environment, or society. As you learn about quadratic modeling, you will explore the relationships between the algebraic form of a quadratic function model and the situation it describes. You will use technology as a tool to solve quadratic equations and to understand how the graphs of quadratic functions relate to their algebraic form. The work for this badge builds a foundation for the future study of topics such as polynomials and complex numbers. Modeling with quadratic functions is useful for careers in a variety of fields, like engineering, military, law enforcement, astronomy, car manufacturing, and agriculture.

Suggested prerequisites for this badge: M102 Modeling with Linear Functions and Equations.

This badge is suggested as a prerequisite for: M201 Function Concepts; M202 Rational Exponents and Complex Numbers; M203 Polynomial and Rational Expressions, Functions, and Equations; M204 Exponential and Logarithmic Functions and Equations.

M104 Modeling with Functions of Exponential Type

What will the money in your savings account be worth in 30 years? How does a coroner determine the time of death? Nature, society, business, and everyday life are full of situations in which a certain quantity depends on other measurable quantities. Function models enable us to describe, analyze, optimize, and predict what can happen in these situations. In M104 Modeling with Functions of Exponential Type, you will create function models in situations where a quantity changes at a rate that is proportional to its value. You will solve problems and reason about situations of exponential growth and decay, such as wildlife populations, financial investments and depreciation, bacterial growth, radioactivity, internet usage, or the popularity of fads. As you learn exponential modeling, you will make sense of the relationships between the algebraic form of an exponential function model and the situation it describes. You will use technology as a tool to solve exponential equations and to understand how the graphs of exponential functions relate to their algebraic form. The work for this badge builds a foundation for the future study of topics such as rational exponents and logarithmic functions. Modeling with exponential functions is useful for careers in a variety of fields, like economics, biology, sound engineering, and statistics.

Suggested prerequisites for this badge: M102 Modeling with Linear Functions and Equations.

This badge is suggested as a prerequisite for: M201 Function Concepts; M202 Rational Exponents and Complex Numbers; M203 Polynomial and Rational Expressions, Functions, and Equations; M204 Exponential and Logarithmic Functions and Equation

M111 Modeling with Data: One-Variable Measurement Data

Some questions have a precise answer. For example, “How many years old is my little brother?” Other questions do not lead to a single answer, but to a set of data. In particular, some important questions about the world around us involve one-variable measurement data. This includes questions such as: “How much carbon dioxide is emitted daily in New York City?” or “Does life expectancy in one country differ meaningfully from that in another country?” By allowing us to analyze such questions, the tools and methods of statistics with one-variable measurement data can help us better understand our world, address the challenges of our time, and advocate for change. In M111 Modeling with One-Variable Measurement Data, you will pose and analyze meaningful statistical questions that yield one-variable measurement data. You will strategically use data displays and quantitative methods to draw conclusions, gain insight into the situation, and generate new questions. As you learn modeling with one-variable measurement data, you will use technology to create and analyze histograms and other visual displays. You will summarize data sets with measures of center, spread, and reason interpreting the meaning of differences in center, shape, and spread. Finally, you will model data with normal distribution and estimate population percentages. The work for this badge builds a foundation for the future study of topics such as modeling with two-variable data, statistical inference, and data science. Modeling with one-variable measurement data is useful for careers in a variety of fields, like statistics, economics, biology, and computer science.

Suggested prerequisites for this badge: understanding of fractions, decimals, and percentages; comfort with using formulas.

M112 Modeling with Data: Two-Variable Measurement Data

Some important questions about the world around us involve possible relationships between two variables. For example, we might want to know not only how life expectancy and poverty rates differ from one country to another, but also how national life expectancy correlates with national poverty levels. By allowing us to analyze such questions, the tools and methods of statistics with two-variable measurement data can help us better understand our world, address the challenges of our time, and advocate for change. In M112 Modeling with Two-Variable Measurement Data, you will pose and analyze meaningful statistical questions that yield two-variable measurement data. You will reason with scatter plots, equation models, and quantitative methods to draw conclusions, gain insight into each situation, and generate new questions. As you learn modeling with two-variable measurement data, you will use technology strategically to create and analyze scatter plots to investigate patterns of association between two measured quantities, creating and reasoning with linear equation models where appropriate. The work for this badge builds a foundation for the future study of topics such as statistical inference and data science. Modeling with two-variable measurement data is useful for careers in a variety of fields, like statistics, psychology, biology, and medicine.

Suggested prerequisite for this badge: M101 Linear Equations: Concepts and Skills.

M113 Modeling with Probability

What makes a game fair? How can we predict the likelihood of a candidate winning an election or a team winning a championship? Modeling with probability is a powerful way to gain insight into the world around us and answer questions like these. Nature, society, business, and everyday life are full of situations involving uncertainty and randomness. Probability models enable us to go beyond guesswork and handle these situations quantitatively. In M113 Modeling with Probability, you will calculate and estimate probabilities by using data. You will make sense of and interpret these results to make predictions, analyze strategies, and inform decisions. As you learn modeling with probability, you will strategically utilize tools like tables, tree diagrams, counting techniques, and the rules of probability. Using technology, you will simulate random processes, approximate probabilities, interpret results, and make appropriate decisions about everyday events. Constructing and interpreting two-way frequency tables to determine if events are independent, approximating conditional probabilities, and calculating the expected value to make informed decisions are also important topics of this badge. Modeling with probability is useful for careers in a variety of fields, like meteorology, risk management, nursing research, public policy, sports, and finance.

Suggested prerequisites for this badge: concepts of fractions, ratio, and percentages.

This badge is suggested as a prerequisite for: M211 Data Management and Visualization.

M151 Modeling with Geometry

How much aluminum is needed to build a prototype robot? Which design for a new cell phone case will have the lowest cost? Modeling with geometry is a powerful way to gain insight into the world around us and solve design challenges. Whether we are laying out a backyard flower bed, advocating for a new neighborhood park, or drafting a blueprint for a building, geometric models hold great power to help us ask and answer questions about the physical world. In M151 Modeling with Geometry, you will apply concepts of measurement to make sense of real-world situations while creating models that consider important situational features, such as cost and size constraints, or aesthetic considerations. As you learn about modeling with geometry, you will use technology strategically to model complex objects and scenarios. You will apply principles of length, area, volume, and angle to model geometric measurements and spatial relationships, and apply the Pythagorean Theorem to solve problems in real-world contexts. Modeling with geometry is useful for careers in a variety of fields, like the arts, engineering, and architecture.

Suggested prerequisites for this badge: comfort with middle-school-level problems involving length, area, volume, and angle measure; comfort with—or interest in learning about—writing and solving simple equations to solve problems; comfort with using formulas.

This badge is suggested as a prerequisite for: advanced high school courses in mathematical modeling.

M152 Reasoning and Proof through Congruence

Congruence provides fertile ground for reasoning about geometric figures, allowing us to hone our deductive skills for use in a wide variety of contexts. We can reason with rigid motions to develop criteria for congruent triangles, enabling us to then prove a wide array of theorems, as well as find unknown angle measures and segment lengths in a variety of figures. Specifically, we can use available evidence to argue that two triangles are congruent or that two angles have the same measures. In M152 Geometry: Reasoning and Proof through Congruence, you will construct and critique logical arguments as you explore relationships between geometric figures. You will perform geometric constructions and describe a sequence of rigid motions to prove that one figure is congruent with another, using technology to aid in performing constructions and transformations. Acquiring the skills of geometric reasoning in these areas leads to the understanding of analytical approaches to geometry and opportunities to apply logical reasoning in fields beyond mathematics. Congruence is useful for careers in a variety of fields, like mathematics, graphic design, animation, art, and physics.

Suggested prerequisites for this badge: basic understanding of angles, triangles, and quadrilaterals.

M153 Reasoning and Proof through Similarity

Similarity provides fertile ground for reasoning about geometric figures, allowing us to hone our deductive skills for use in a wide variety of contexts. We can build on reasoning with rigid motions to include dilations in order to develop criteria for similar triangles, enabling us to then prove a wide array of theorems that allow for determining unknown angle measures and segment lengths in a variety of figures, and for creating animated models using technology. In M153 Geometry: Reasoning and Proof through Similarity, you will construct and critique logical arguments as you explore relationships between geometric figures. You will perform geometric constructions and describe a sequence of transformations to prove that one figure is similar to another, using technology to aid in performing transformations. You will also test and apply theorems to reason about relationships between figures to prove similarity, as well as to construct visual representations of images your mind creates using segments, angles, triangles, or quadrilaterals. Acquiring the skills of geometric reasoning in these areas leads to the understanding of analytical approaches to geometry and opportunities for applying logical reasoning in fields beyond mathematics. Similarity is useful for careers in a variety of fields, like mathematics, graphic design, animation, art, and physics.

Suggested prerequisites for this badge: basic understanding of angles, triangles, and quadrilaterals.

M154 Coordinate Geometry

How do pilots keep their aircraft on course? How are algebra and geometry related? Coordinate Geometry, also known as analytic geometry, establishes a connection between geometric curves and algebraic equations. In M154 Coordinate Geometry, you will reason and use algebra to do geometry, and use geometry to do algebra. As you learn coordinate geometry, you will investigate and make sense of conic sections, produce equations and graphs for circles, ellipses, and hyperbolas, and solve simple systems algebraically and with technology systems consisting of linear and quadratic equations. Using coordinates to answer questions about shapes in the coordinate plane will strengthen your ability to compute perimeters of polygons and areas of triangles and rectangles and to determine the equation of a line parallel or perpendicular to a given line. You will strategically use technology to explore the graphs of geometric curves and the related algebraic equations. For example, when designing an architectural arch, you might use graphing technology to manipulate an equation and see the result. Coordinate Geometry is useful for careers in a variety of fields, like computer graphics in games and films, space science, aviation, and engineering.

Suggested prerequisites for this badge: basic understanding of angles, triangles, quadrilaterals, and circles; M101 Linear Equations: Concepts and Skills.

This badge is suggested as a prerequisite for: Precalculus and Calculus.

M155 Right Triangle Trigonometry

How can you estimate the height of a tree? How can you create an accurate blueprint for a new building? How can similarity shed light on relationships between angles and sides in right triangles? Trigonometry is a powerful tool that illuminates geometric relationships and helps us solve problems involving right triangles. In M155 Right Triangle Trigonometry, you will develop an understanding of what trigonometric ratios are and how they can be used to solve real-world and mathematical problems. You will use technology as a tool to calculate trigonometric ratios and find missing measurements in diagrams. Right triangle trigonometry is useful for careers in a variety of fields, like the sciences, engineering, forestry, criminology, and architecture.

Suggested prerequisites for this badge: M153 Reasoning and Proof through Similarity.

M201 Function Concepts

Are you ready to deepen your knowledge of functions? These are versatile modeling tools that allow us to represent any situation in which a change in one measurable quantity results in a change in a related measurable quantity. Understanding functions allows us to create, analyze, and interpret them in a variety of contexts. In M201 Function Concepts, you will extend and deepen your mathematical understanding of linear, exponential, and quadratic functions. You will understand sequences as functions and explain relationships between explicit and recursive representations. Additionally, you will reason and generalize ideas about transformations of functions, using technology as an aid to analyze your graphs while also exploring ideas of domain and range and constructing inverse functions. Understanding function concepts sets the stage for further work with more complicated functions, including polynomial, rational, exponential, and logarithmic functions. Function concepts are useful for careers in a variety of fields, like the sciences and engineering.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

M202 Rational Exponents and Complex Numbers

How do weather forecasters create models to determine if an umbrella is needed that day? Rational exponents play an important mathematical role by connecting powers and roots, as well as in real-world situations like depreciation and inflation. Complex numbers ensure that every algebraic equation has a solution, and they make the work of designing Wi-Fi equipment, electric cars, and wind turbines easier. In M202 Rational Exponents and Complex Numbers, you will complete your study of the real number system and extend your study of numbers beyond it to the complex number system. As you make sense of rational exponents and complex numbers, you will evaluate and simplify numerical and algebraic expressions involving radicals and rational exponents, solve simple rational and radical equations in one variable, and add, subtract, and multiply complex numbers. Additionally, you will solve quadratic equations with real coefficients containing non-real solutions and that represent complex numbers on the complex plane in rectangular form. Complex numbers are beneficial in describing many complex situations, such as the laws of electricity and electromagnetism. Both rational exponents and radical expressions are useful for careers in a variety of fields, like architecture, carpentry, electrical engineering, finance, and masonry.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

M203 Polynomial and Rational Expressions, Functions, and Equations

How do aerospace engineers determine the acceleration of a rocket? How do astronomers calculate the distance of a new star and the Earth? If you add, subtract, multiply, or divide two numbers, the answer will be another number. If you add, subtract, multiply, or divide two algebraic expressions, the result will be another algebraic expression: a polynomial or rational expression, to be precise. In this way, advanced algebra generalizes the familiar process of arithmetic. In the same way that arithmetic is useful for solving simple world problems, polynomial and rational functions are useful for modeling a variety of situations in contexts such as meteorology, economics, and financial planning. In M203 Polynomial and Rational Expressions, Functions, and Equations, you will reason about and explore the interplay between algebraic expressions and the functions they define. As you learn these mathematics, you will make sense of the properties used to manipulate, add, subtract, multiply, and factor polynomial expressions. Understanding how factors and zeros of polynomials are related will aid in graphing polynomial and rational functions and solving polynomial equations by factoring and using given factorizations. Using technology tools such as computer algebra systems (CAS) will be beneficial for making sense of the features of the graph and how the graph relates to the algebraic form. Polynomials and rational expressions, functions, and equations are useful for careers in a variety of fields, like aerospace engineering, mechanical engineering, and astronomy.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

This badge is suggested as a prerequisite for: Precalculus and Calculus.

M204 Exponential and Logarithmic Functions and Equations

How are exponents and logarithms related? Some quantities can vary by many orders of magnitude, like world population over time, housing prices, or the energy released by earthquakes. In M204 Exponential and Logarithmic Functions and Equations, you will gain the tools needed to understand and solve problems in situations like these. As you learn these mathematics, you will understand that exponential and logarithmic functions are inverse functions, gain the foundation to make sense of their graphs and characteristics, comprehend the rules for manipulating expressions involving exponents and logs to solve equations, and learn the relationship between base *e*, its exponential function, and the natural logarithm. Exponential and logarithmic functions are important nonlinear functions and are useful for careers in a variety of fields, like nuclear and internal medicine, forensic science, and finance.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

This badge is suggested as a prerequisite for: Precalculus and Calculus.

M205 Trigonometric Functions

Why do some real-world phenomena produce graphs shaped like waves? How can trigonometric functions be used to model the rising and falling of the ocean waters each day? Using radian measure, mathematicians have been able to define trigonometric functions, an intriguing function family with unique properties that can also help us to solve problems and model phenomena. In M205 Trigonometric Functions, you will explore the unit circle and develop understanding of how radians are defined. You will explore the properties of trigonometric functions, using graphing applications to aid in your analyses, and find opportunities to use these functions to model real-world situations. Trigonometric functions are useful for careers in a variety of fields, like the sciences, engineering, oceanography, and gaming.

Suggested prerequisites for this badge: M155 Right Triangle Trigonometry.

M211 Data Management and Visualization

How might a social media company's data about teen social media use differ from teens' self-reported hours of social media use? What part of a story is told by data? How is data generated or collected? What kind of measurement can give you insight into an unanswered question? Data can sometimes be very messy, and its collection requires ethical consideration. In M211 Data Management and Visualization, you will explore the first steps a data scientist takes when dealing with univariate, bivariate, and multivariate data. You will organize data into rows and columns and learn how to deal with missing values. You will practice representing and describing data, transforming it as needed for a desired predictive modeling procedure. You will make decisions about what visual representation is best depending on the type of data you have. Using data visualization, you will consider what story you can tell from the data. After gathering data ethically, you can expect to use technology to organize, clean, and represent it in a meaningful way. Data management and visualization are useful for careers in a variety of fields, like data analytics, marketing, programming, and investigative journalism.

Suggested prerequisites for this badge: concepts of addition, subtraction, multiplication, and division; ratio concepts; solving problems involving percentages; M113 Modeling with Probability.

This badge is suggested as a prerequisite for: M212 Predictive Modeling; M213 Statistical Error and Predictive Model Validation.

M212 Predictive Modeling

Can the number of people in a household be used to predict water usage? Can you use technology to predict whether or not you will like a song before hearing it? Can the amount of caffeine consumed predict pulse rates amongst your classmates? Are there meaningful categories in your data that technology can detect? In M212 Predictive Modeling, you will master the skill of asking statistical questions, deciding what kind of model will answer your question, and a few types of supervised and one or two types of unsupervised learning methods that will hopefully help you answer your questions. Supervised learning methods rely on the idea that an output can be predicted from one or multiple inputs with some amount of error. Such methods may include Linear Regression, Multiple Regression, Logistic Regression, Classification, or Decision Trees. Unsupervised learning methods allow technology to recognize patterns in the data or categorize the data to make sense of it. Such methods may include Cluster Analysis, Dimensionality Reduction, and Feature Selection. You will distinguish between correlation and causation when a relationship appears to exist between variables. Predictive modeling is useful for careers in a variety of fields, like actuarial science, engineering, public health, and environmental sciences.

Suggested prerequisites for this badge: M211 Data Management and Visualization.

This badge is suggested as a prerequisite for: M214 Statistical Error and Predictive Model Validation.

M213 Bayesian Reasoning and Probability Theory

Does the probability of becoming a professional athlete differ based on height? Does the probability of seeing yourself represented in media differ based on skin tone? Does the probability of flipping “tails” using a fair coin decrease if you just flipped “tails” three times in a row? Outcomes may differ depending on what events have occurred prior, which means that the probabilities within a sample space could be different from what they would be without these prior events. In M213 Bayesian Reasoning and Probability Theory, you will learn the meaning of independence and how it relates to Bayes’ Rule by learning and using multiple other probability rules. The use of two-way tables will help break down some of these concepts. You will learn that probability, by definition, is the expected proportion of times a desired event occurs. Using two-way tables and probability rules, you will calculate probabilities and decide if events are independent. You will recognize that these probabilities may change depending on prior conditions. You will strategically use tools like tables, tree diagrams, counting techniques, and the rules of probability. Using technology, you will simulate random processes, approximate probabilities, and interpret results. Bayesian reasoning and probability theory are useful for careers in a variety of fields, like biology, epidemiology, and risk management.

Suggested prerequisites for this badge: M211 Data Management and Visualization.

This badge is suggested as a prerequisite for: M214 Statistical Error and Predictive Model Validation.

M214 Statistical Error and Predictive Model Validation

How accurate is a model for predicting your shoe size based on your height? What tools can you use to assess how far off these predictions are? What are the pros and cons of having less error in a predictive model? It is very rare to create a perfect predictive model, so how do you know whether to trust the models you create? In M214 Statistical Error and Predictive Model Validation, you will explain and quantify variation and its sources while mastering multiple methods of model validation, uncertainty, and comparison to inform decisions. You will utilize technology to practice statistical techniques to improve your predictive models such as F-tests, sensitivity analysis, transforming variables, adding or excluding variables, ANOVA, residual plots, normalizing variables, or cross validation. You will learn about the balance between bias and variance, and that a model with low variance does not necessarily equate to the best possible model. Your ultimate goal will be to answer the question “What might happen in the future?” based on prior knowledge—e.g., your data. You may use this information to diagnose a problem and draft its solution. Statistical error and predictive model validation are useful for careers in a variety of fields, like actuarial science, engineering, public health, and environmental sciences.

Suggested prerequisites for this badge: M212 Predictive Modeling; M213 Bayesian Reasoning and Probability Theory.

This badge is suggested as a prerequisite for: M215 Inference and Making Conclusions.

M215 Inference and Making Conclusions

What is the true mean height of all students at your school? What is the true proportion of Hershey’s Kisses that will land on their flat side when tossed on a table? Has the true mean carbon dioxide level of the atmosphere increased significantly over the last five years? Does a new brand of sunscreen perform significantly better on average than an old one? In M215 Inference and Making Conclusions, you will master the logic of inference, learn a couple of inference procedures, and make decisions after weighing the risks associated with them. You will calculate and interpret P-values with technology. The confidence intervals you calculate will help inform choices about significant changes. You will learn what Type I and Type II errors are, as well as evaluate which error is worse in the context. You will use bootstrapping to improve the accuracy of your inference procedures and will make decisions based on how a study is designed and how small your P-value is. Inference and conclusion-making are useful for careers in a variety of fields, like psychology, biology, advertising, and manufacturing.

Suggested prerequisite for this badge: M214 Statistical Error and Predictive Model Validation.

High School Math Badge Frameworks

M100 Quantitative Reasoning

Badge Catalog Description

How long will it take to save up for a new phone? Is affordable housing distributed fairly? Math helps us answer questions like these and reason effectively in a variety of real-world situations, including designing a building, interpreting survey results, or comparing costs of credit card offers.

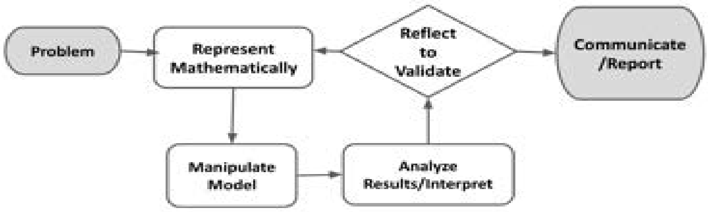
In M100 Quantitative Reasoning, you will make sense of and solve real-world problems by strategically using tools like proportionality and percentages, geometric measurement, and data displays. Using technology as an aid for calculations, you will reason about quantities in different units and use formulas to solve problems. Quantitative reasoning is a critical tool that can lead to more sophisticated modeling with functions and geometry. It is also the bedrock of the mathematics of our daily lives and widely applicable to nearly every professional field.

Suggested prerequisites for this badge: concepts of addition, subtraction, multiplication, and division; fraction and decimal concepts; understanding of positive and negative numbers.

Badges to try next: M111 Modeling with One-Variable Measurement Data; M101 Linear Equations: Concepts and Skills; M102 Modeling with Linear Functions and Equations; M151 Modeling with Geometry.

The M100 badge calls on students to engage with foundational concepts from algebra, measurement, statistics, and proportional reasoning in ways that connect to the world around them. This badge integrates aspects of the mathematical modeling cycle, allowing for content design that produces relevant and authentic tasks.

The learning expectations for M100 involve aspects of the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards. (Adapted from Common Core Standards Writing Team, 2019, Modeling, K-12, p.6)

A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
It involves

1. Identifying variables in the situation and selecting those that represent essential features.
2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
3. Analyzing and performing operations on these relationships to draw conclusions.
4. Interpreting the results of the mathematics in terms of the original situation.
5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72)

The way that students engage in modeling in the M100 badge draws on a variety of areas of mathematics and includes using mathematics from earlier grades *at a level of sophistication appropriate for high school students*. The following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M100 badge.

M100 Content and Practice Expectations

100.a	Identify quantities of interest for modeling purposes.
100.b	Reason with units in problems, formulas, and data displays to solve problems.
100.c	Interpret simple numeric and algebraic expressions that arise in applications in terms of the context.
100.d	Interpret equations, tables, and graphs that arise in applications involving proportional relationships.
100.e	Interpret data displays, such as line plots, histograms, and box plots in terms of the context.

100.f	Solve real-world problems involving distances, intervals of time, liquid volume, mass, money, and other situations where operations with fractions and decimals are appropriate.
100.g	Solve real-world problems involving area and perimeter of polygons, as well as surface area and volume of prisms and pyramids.
100.h	Solve real-world problems involving quantities in a proportional relationship, including scale drawings.
100.i	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Learning Principles

In M100 Quantitative Reasoning, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting “doing mathematics,” but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M100

The M100 badge ties together concepts from several different areas of mathematics, including fraction and decimal operations, measurement, proportional relationships, and quantitative data, in service of authentic problem-solving. Typical instruction in these areas might emphasize memorization of formulas or rules for computations, and focus time on students performing computations absent relevant or real-world contexts. However, M100 is most fundamentally about using mathematics to solve contextual problems, and as such, students should:

- regularly encounter real-world tasks that require them to make sense of situations and relate expressions, equations, and data displays back to the contexts they represent (LP 1).
- engage with aspects of the modeling cycle (see above), including identifying quantities of interest, representing situations with equations, expressions, and data displays, and choosing an appropriate level of accuracy when reporting quantities (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics of interest, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their solution methods (LP 2).

Spending time manually computing and generating tables or graphs should not be a focus of this badge. In fact, the opposite is true. In M100, students should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models. For example, students should:
    - make sense of problem situations, discussing and identifying quantities of interest.
    - explain the real-world meaning of expressions, equations, formulas, and data displays, including the units, scales, variables, and parameters in the expressions.
    - compare two different representations of the same context.
    - use models to justify a claim about a real-world context (LP 3).

- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Applications for M100 are plentiful, since fraction and decimal operations, data displays, measurement concepts, and proportional relationships appear frequently, not only in everyday life, but also in a variety of academic disciplines. Typical applications might include problems involving travel, design, and business. In M100, students should also be given opportunities to understand and reflect on the ways that these mathematical concepts and skills can lend themselves to meeting global economic, social, and environmental challenges (LP 6). Some contexts to consider include:

- climate change and other environmental concerns.
- disparate policy impacts on racial and gender identities.
- wealth inequality.
- equitable housing.

Students should be given frequent opportunities to use spreadsheets and graphing software to allow for a focus on understanding, rather than computation or symbolic manipulation. Students should write spreadsheet formulas to aid in solving design problems with area, surface area, and volume (LP 7).

Evidence of Learning

In M100 Quantitative Reasoning, students' evidence of learning is demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.
Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
100.a: Identify quantities of interest for modeling purposes.	i. make sense of a situation and choose key quantities. Explore the relationship between those quantities to answer a question of interest.
100.b: Reason with units in problems, formulas, and data displays to solve	i. determine an appropriate scale for a graph or data display.

Content and Practice Expectations	Indicators Choose an artifact where you...
problems.	ii. manipulate units as part of the problem solving process.
	iii. explain how the units (including derived units, where appropriate) represented in a solution relate to the problem context, formula, and/or data display.
100.c: Interpret simple numeric and algebraic expressions that arise in applications in terms of the context.	i. select two different expressions that illustrate something interesting or important about the context you are modeling and describe what these expressions tell you about the context.
	ii. describe the contextual meaning of a number or variable of interest in an expression.
100.d: Interpret equations, tables, and graphs that arise in applications involving proportional relationships.	i. select two different representations (equations, tables, or graphs) that illustrate something interesting or important about a proportional relationship you are modeling and describe what these representations tell you about the context.
	ii. describe the contextual meaning of a number, variable, or coordinates in an equation, table, or graph of interest.
100.e: Interpret data displays, such as line plots, histograms, and box plots in terms of the context.	i. select two different data displays that illustrate something interesting or important about a real-world situation and describe what these representations tell you about the context.
	ii. describe the contextual meaning of a key component of a data display (e.g., describe the meaning of one bar on a histogram).
100.f: Solve real-world problems involving distances, intervals of time, liquid volume, mass, money, and other situations where operations with fractions and decimals are appropriate.	i. write a numeric or algebraic expression to model a situation.
	ii. use a numeric or algebraic expression model to compute values and then interpret the values in the context of the problem.
100.g: Solve real-world problems involving area and perimeter of polygons, as well as surface area and volume of prisms and pyramids.	i. create an equation, table, or graph to model a context involving area and perimeter of polygons and/or surface area and volume of prisms and pyramids.
	ii. use an equation, table, or graph to solve a problem.
	iii. interpret the values of the solution in terms of the context.

Content and Practice Expectations	Indicators Choose an artifact where you...
100.h: Solve real-world problems involving quantities in a proportional relationship, including scale drawings.	i. create an equation, table, or graph to model a proportional relationship.
	ii. use an equation, table, or graph to solve the problem.
	iii. interpret the quantities of the solution in terms of the context.
100.i: Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.	i. explain the level of accuracy chosen when answering a contextual question.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M100 Quantitative Reasoning (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a task like this:

Ayhan came across his great grandmother's recipe for lemon cake. He mixed all the ingredients together in his 3 quart mixing bowl and noticed the batter almost filled up the bowl.

Now he has to determine which baking pan to use. He has a set of pans that include the following:

- *A loaf pan that measures 8.5 inches by 4.5 inches by 2.5 inches*
- *A square pan that measures 8 inches by 8 inches by 2 inches*
- *An oblong pan that measures 13 inches by 9 inches by 1.75 inches*

Which pan should he use?

Sample Learning Experience

Launch this prompt by asking students to share what they know about baking. Do they have any recipes that are really important to them personally or to their families? Chart ideas that surface so that students can reference later. Allow students to search baking tips as needed.

Give students some quiet work time to consider their approach and response. Monitor students as they work. Look for students who make simplifying assumptions, as well as students who use expressions and equations to reason about this prompt. Look for students who recognize baking pans should not be filled to the brim, since the batter will spill over in the baking process. Select students with varied approaches to share their thinking with the class.

The following questions can be used to engage students in each other's work:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*

Be sure to make visible any simplifying assumptions students make, as well as any insights offered about baking that come from student experience and research. Prompt students to explain the meaning of the numbers and any other relevant quantities used in their model.

Once different approaches have been shared, students could be given an opportunity to revise their work.

In this example, students are:

- given a cognitively demanding task. It requires that students model with geometry using an authentic real-world situation (LP 1).
- set up to have social interactions, as they vary working styles, working both individually and in partnership with others, as well as share and compare their methods (LP 2).
- deepening their understanding of the conservation of volume, considering the geometric attributes of everyday objects, and looking for the relationship between various measures of volume (LP 3).
- able to bring in lived experiences grounded in baking and use that knowledge to solve mathematical problems (LP 4).
- able to develop an appreciation of mathematics as a human endeavor, as students see simple things like recipes as both mathematical and generational (LP 5).
- able to engage with mathematics in ways that authentically involve real-world situations as well as have the opportunity to recognize that mathematical modeling is something we do even when it comes to finding the right container to bake a cake (LP 6).
- able to use technology as a tool, as they use computational aids such as software or calculators to aid in making calculations (LP 7).

Example 2

Students are given a problem like this:

The purpose of this task is to build an understanding of how app developers go about designing user-friendly tools. The sponsors of a walk-a-thon to raise funds for cancer research would like to offer an app for participants to use before, during, and after the event.

Your job is to (1) draft a flowchart that outlines how a fitness app of your design would work for this event, (2) design a table, using spreadsheet software, that operates on key quantities of your model to answer a question that might be of interest for a participant, or a program, using your favorite tech tool, and (3) prepare a presentation outlining your proposal.

Part A: Making sense of the ask

1. *What sort of things do you anticipate people do to prepare for an event like a walk-a-thon?*
2. *What sort of questions do you imagine people will want answered when using the app?*
3. *What sort of information is needed to answer these questions? What are the mathematical relationships between the various quantities you've identified and the questions folks might have?*
4. *What additional factors do you think need to be considered when creating an app that helps folks prepare for an event like this walk-a-thon?*

Part B: Exploring existing apps and drafting the flow of your app

5. *Take some time to learn more about popular fitness apps currently available. Try out free apps [*The 9 Best Free Fitness Apps [2022] - Boomfit*] or interview people you know who use fitness apps to learn more about this product.*
    a. *What new understandings do you have about how these apps function?*
    b. *What quantities do you see displayed for the user? How are relationships between quantities used to provide information that is useful to the user?*
    c. *What new understandings about the relationships between the quantities identified in Part A do you have?*
6. *Based on your current understanding of apps, preparing for physical activity, the quantities you identified, and the relationships between those quantities, draft a flowchart of how the app you envision would work.*

Part C: Designing a table and second draft of app flowchart

7. *Use spreadsheets or other tech tools to design a table or program that could be used within your app to take information from the user to produce quantities of interest. Describe the mathematical relationships your table is designed to use.*
8. *Test out your table or program with people you know. What new insights do you have?*

Part D: Reflecting, revising, and synthesizing the learning

9. *What worked well about the table or program you designed? What did not? How would you modify what you have? Create a sample screenshot of how you envision this aspect of your app to work. Describe the mathematical relationships in play. What are the limitations of the model you designed? What ideas for future development do you recommend?*
10. *Revisit the flowchart you drafted earlier in the process. What modifications would you make? Make a third draft of the flow of your app.*
11. *What surprised you about this process?*

Extending the Learning *(Optional)*

1. *Explore the work of software engineers, architects, and developers.[*Principal Software Engineer vs Architect vs Developer | NCube *]What elements of what you accomplished fit under each of these roles? What new insights do you have about this field?*
2. *One approach to app development is to build apps that have a wide range of applications. Having the ability to reuse core aspects of a program for different functions gives programs the ability to increase their earning potential. Consider the following group of possible participants. What would you add or modify to your app to make this something every person could continue to use beyond the walk-a-thon event?*

Name	Age (years)	Height (feet)	Weight (lbs.)	Activities	Workout 1	Workout 2	Workout 3
Nico	15	5’10”	120	Cross country running, track and field	5k run 18:36 min.	5k run 20:01 min.	5k run 17:46 min.
Raj	44	6’0”	175	Tennis, swim	Tennis 2 hrs.	Tennis 2 hrs.	Tennis 2 hrs.
Maura	51	5’4”	140	Walking, dancing	Walk 30 min.	Dance 1 hr.	Walk 40 min.

Sample Learning Experience
As students work through the different parts of this task, they could be prompted to discuss their approaches in small groups or partnerships. The teacher can select student work strategically and pose questions such as these:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*
- *What additional questions could be answered about this context using some of the models we've seen? Are some models more helpful than others in answering these additional questions?*

Following the discussion, students are given an opportunity to reflect on and revise their work. As students share their work: elevate key mathematical aspects of their work, such as the relationships between the quantities they've chosen; bring to light any underlying assumptions in play; and as appropriate, probe about degree of accuracy in figures presented. Some questions are as follows: *when and how does your model round? To what degree? Does that make sense? Why or why not?*

Provide access to various tools as students put together a presentation of their designs. Once presentations have been shared, students could be given an opportunity to revise their work.

Additional Resources
Part B: Exploring existing apps and drafting the flow of your app
- Fitness App Development: Types, Features, and Costs | Mobindustry
- Best Fitness Apps Of 2022 – Forbes Health
- Best Fitness and Exercise Apps
- The Best Fitness Apps of 2022 - SI Showcase
- Best Workout Plan Apps to Learn From | Mobindustry

Part C: *Designing a table and second draft of app flowchart*
- Computer Science: Algorithms
- Computer Science: Sequences, Selections, and Loops
- Software Design Patterns in C++ & UML — Hi I'm Simon

In this example, students are:
- given a cognitively demanding task. It requires students to develop a conceptual model for an app that utilizes user data as the basis of mathematical computations for the purposes of predicting quantities of interest (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- deepening their understanding of identifying key quantities and mathematical relationships needed to make useful models for fitness choices (LP 3).
- able to explore areas of interest and develop a lens for aspirations, as they consider the work of programmers, developers, and architects (LP 4).

- able to develop an appreciation of mathematics as a human endeavor, as students see how the application of ratio reasoning is used to make fitness choices (LP 5).
- able to engage with mathematics in ways that authentically involve real-world situations as well as have the opportunity to develop a conceptual model for a fitness app (LP 6).
- able to use technology as a tool, as they use computational aids such as software or calculators to aid in making calculations (LP 7).

Example 3

Students are given a task like this:

The United States Environmental Protection Agency (EPA) "estimates that more food reaches landfills and incinerators than any other single material in our everyday trash, constituting 24 percent of the amount landfilled and 22 percent of the amount combusted with energy recovery." [source: Regional Resources to Reduce and Divert Wasted Food Across the United States | US EPA]

A study completed by The Drawdown Review: Climate Solutions for a New Decade *shared "composting organic waste versus landfilling can reduce more than 50% of carbon dioxide-equivalent greenhouse gas emissions, for a total of 2.1 gigatons between now (2020) and 2050 if climate change is curbed to a 2 degree Celsius rise in the average global temperature." [source: The ComPOSTer: How much can composting help in solving the climate challenge? [UPDATED]]*

Your task is to develop a public service announcement (PSA) in the form of an infographic or advertisement (suitable for radio, social media, or television) to inform your community about ways they can help minimize the amount of food and organic waste that goes into landfills.

Your PSA should include information that informs your community about:
- *waste trends in your local area*
- *the long-term impact of current trends*
- *ways to change current trends for positive impact*

Your PSA should:
- *identify relevant quantities*
- *use appropriate graphical displays of these quantities*

Sample Learning Experience
Students can be introduced to this topic in a variety of ways. One way to raise awareness and create space for students to make sense of waste management is to engage students with Student Achievement Partners' Modeling Task: Super Fill. Build on this activity by inviting students to use what they now understand about landfills to explore additional considerations. Articles by the US Environmental Protection Agency, linked below, are a good place to start.

Another option is to begin by giving students time to explore resources. A sample list is provided in Part 1 of Additional Resources. Structure this time so that students have time to read and summarize findings. This can be done individually or in pairs. Encourage students to identify questions that surface as they read through the materials. Invite students to share findings and questions with the class, and make a public record of questions that surface.

Encourage students to reflect on the preliminary presentations and make note of how they want to move forward on this task. Some questions to consider are as follows:

- *What resonates with you?*
- *What questions pique your interest and drive you to learn more?*
- *What information will be helpful as you prepare to make your PSA?*
- *What questions will help focus your PSA so that community members take your advice?*

Give students time to conduct further research to answer some of their questions. Encourage students to explore infographics and other materials that have been developed to inform the public. Select and facilitate a discussion inviting students to make sense of the quantities of interests used in existing materials, examples of which can be found under Part 2 of Additional Resources below. Some questions to pose are as follows:

- *What does the quantity_____ represent in this material?*
- *Why do you think the author/creator chose to highlight that quantity?*
- *How does that quantity relate to what you already know about this topic?*
- *Where do you think they got these figures?*
- *Where/how might you look for information relevant to our context?*
- *What have you learned from studying existing materials that you want to apply to your work?*

Give students time to conduct further research to answer some of their questions. As students develop their PSA, provide opportunities for students to share progress with classmates. Consider the following questions to facilitate discussion:

- *What's promising about the planned PSA?*
- *What questions surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Give students time to conduct further research and revise their PSA. As students finalize their PSA, share products with classmates and at your site. Explore ways to partner with local businesses and radio and television stations in order to broadcast student work. Leverage social media outlets and electronic applications that your site has available.

Additional Resources

Part 1: Making sense of waste management

- Wasted Food Programs and Resources Across the United States | US EPA
- National Overview: Facts and Figures on Materials, Wastes and Recycling | US EPA
- The ComPOSTer: How much can composting help in solving the climate challenge? [UPDATED]

Part 2: Making sense of infographics

- Food Waste — Information is Beautiful
- Consumers | USDA

Part 3: Making sense of composting efforts

- Composting efforts gain traction across the United States - The Washington Post
- Beginner's Guide to Composting | One Small Step | NowThis
- Composting for Beginners | The Dirt | Better Homes & Gardens
- 6 Different Ways To Compost, No Matter Where You Live
- Why is composting so hard in the United States? | Greenbiz

- https://achievethecore.org/content/upload/Modeling-Task_Landfill_Final_2_6_20.pdf

In this example, students are:

- given a cognitively demanding task. It requires that students make sense of the waste management process, human impact on climate change, and the quantities involved in communicating these relationships (LP 1).
- set up to have social interactions, as they develop, share, and compare their approaches (LP 2).
- deepening their understanding of number, quantities, and mathematical modeling in order to design a product that informs community members. As students make sense of these phenomena and apply their understanding of math concepts, like proportional reasoning, students have the opportunity to represent their findings in a way that educates community members (LP 3).
- able to make visible the ways mathematics is used to quantify and describe the actions and impact of waste and waste management, as they explore quantities of interest within this context (LP 4).
- able to develop a sense of belonging and ability to make an impact in their communities as they share their products through various outlets. This process offers students opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- able to engage with mathematics in ways that authentically involve real-world situations as well as have the opportunity to recognize simple cases of mathematical modeling (LP 6).
- able to use technology as a tool, as they use computational aids such as software or calculators to aid in making calculations (LP 7).

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M101 Linear Equations: Concepts and Skills

Badge Catalog Description

How does the graph of a system of linear equations in two variables show the solution to that system? How is the perimeter of a rectangle related to its side lengths? Equations and inequalities are powerful tools allowing us to communicate mathematical relationships, solve problems, and model phenomena. In particular, linear equations and inequalities are useful for finding measurements in a geometric figure, exploring how constraints on production could impact business profits, or analyzing a myriad of other situations.

In M101 Linear Equations: Concepts and Skills, you will reason about linear equations and develop fluency with graphical and other methods to determine solutions, using technology as an aid. Understanding linear equations and inequalities helps understand quadratic, exponential, and other equation types, as well as modeling and problem-solving in a variety of increasingly complex contexts. Linear equations and inequalities are useful for careers in a variety of fields, like the sciences, medicine and healthcare, business, and engineering.

Suggested prerequisites for this badge: comfort with solving simple equations and problems involving proportional relationships.

This badge is suggested as a prerequisite for: M112 Modeling with Two-Variable Measurement Data; M154 Coordinate Geometry.

The M101 Linear Equations: Concepts and Skills badge puts mathematical reasoning at the center of how students engage with linear equations. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with linear equations and inequalities. According to Standard for Mathematical Practice 7, students “can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects” (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 8). According to the High School Algebra progression, “They are expected to move from repeated reasoning with pairs of points on a line to writing equations in various forms for the line, rather than memorizing all those forms separately” (Common Core State Standards Writing Team, 2013, p. 3). By building on repeated reasoning, students gain insights into the mathematical relationships between quantities in play.

As students engage in M101 Linear Equations: Concepts and Skills, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M101 badge.

M101 Content and Practice Expectations

101.a	Reason about and solve one-variable linear equations and inequalities.
101.b	Solve real-life mathematical problems using linear expressions, equations, and inequalities.
101.c	Analyze and solve two-variable linear equations and pairs of simultaneous linear equations.
101.d	Represent and solve linear equations and inequalities graphically.

Learning Principles

In M101 Linear Equations: Concepts and Skills, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M101

Whereas a typical instructional unit on linear equations and expressions might begin with a focus on students performing computations or manipulating algebraic expressions disconnected from real-world contexts, in M101, students should:

- examine the structure of an equation before manipulating it in order to gain insights about the relationships involved (LP 3).
- use variables to represent quantities to write and solve equations and inequalities in mathematical or real-world situations (LP 3).
- have opportunities to understand and explain why certain equation-solving moves produce equations with the same solution (LP 3).
- recognize that different forms of an equation are helpful for different purposes or for particular contexts (LP 3).
- understand naming a solution of a linear equation or inequality as making a mathematical claim and employ reasoning to justify the solution or its reasonableness (LP 3).
- regularly encounter real-world tasks involving constant rate of change (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, tabular, and graphical (LP 3).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).
- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, algebraic, numerical, graphical. As examples, students should engage with the following:
    - Reason about and explain the real-world meaning of the slope and *y*-intercept.

    - Make connections between the different representations of a situation.
    - Engage with a variety of equations to reason about having one solution, no solution, or an infinite number of solutions.
    - Reason about systems of equations by determining if a system has one solution, no solution, or an infinite number of solutions and extend this knowledge to understand that the solution to a system of inequalities is a region of the plane.
    - Determine the reasonableness of solutions and understand what it means to find the solution to an equation.
    - Explain the relationship between parameters in an equation and the features of its graph (LP 3).
- use open-ended tasks to interpret data, analyze graphs, and reason about the context (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice-sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that equations can authentically represent real-world situations (LP 6).

Often, coursework with linear equations is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use spreadsheets and graphing software to allow for focus on understanding, rather than computation or symbolic manipulation. Students should write spreadsheet formulas that exhibit linear equations (LP 7).
- use various technologies like motion detectors to measure distance vs. time (LP 7).

Evidence of Learning

In M101 Linear Equations: Concepts and Skills, students' evidence of learning is demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
101.a: Reason about and solve one-variable linear equations and inequalities.	i. use variables to represent numbers and write equations and/or inequalities when solving a real-world or mathematical problem.
	ii. demonstrate understanding that a variable can represent an unknown number or, depending on the purpose at hand, any number in a specified set.
	iii. give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions.
101.b: Solve real-life and mathematical problems using linear expressions, equations, and inequalities.	i. solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form while using tools strategically.
	ii. construct simple equations and inequalities to solve problems by reasoning about quantities.
101.c: Analyze and solve two-variable linear equations and pairs of simultaneous linear equations.	i. solve linear equations in two-variables.
	ii. analyze and solve pairs of simultaneous linear equations.
101.d: Represent and solve linear equations and inequalities graphically.	i. explain why the x-coordinates of the points where the graphs of the linear equations y=f(x) and y=g(x) intersect are the solutions to the equation f(x)=g(x).
	ii. find solutions to a system of linear equations in the form y=f(x) and y=g(x) approximately by using their graphs.
	iii. graph the solutions to a linear inequality in two variables as a half plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After	The student's artifact shows evidence of approaching a full understanding of the	The student's artifact demonstrates evidence that they have met the

conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	expectations of the indicator(s).

Annotated Examples M101 Linear Equations: Concepts and Skills (Optional)

The examples that follow are intended to illustrate how learning experiences are used to support students' engagement with the content and practices outlined in this badge. They do not provide comprehensive coverage of those expectations, but rather elevate some of the learning experiences less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a scenario like this:

The math class was asked to solve $4x + 12 < 6x+18$. *One student answered as follows:*

$$4(x + 3) < 6(x + 3)$$
$$4 < 6$$

Therefore, there is no solution.
Decide if you agree or disagree with the student's solution and explain your reasoning.

Sample Learning Experience

Provide time for students to consider the question posed and independently develop a response. Have students write their response including their justification. Next, have partner pairs share their written justification and critique each other's response, allowing for students to defend their response. Provide time for students to revise their written justifications using the information gained during their partner-sharing time. While observing student discussions, identify varying solution methods to be presented during the whole group sharing. Have selected students present their responses—consider using a document camera for students to show actual work—and explain their justification.

Some initial justifications that may surface:

Gonzalo: *"Well if you look at it, you can tell 18 is bigger than 12. And* $6x$ *is always more than* $4x$. *So, that means this is true. So, yeah, 6 is greater than 4. No matter what."*

Nuri: "*I don't agree. The answer is right here.* [Student points to $4 < 6$] *I also see the same thing here* [Student points to expression $x + 3$ on both sides of the inequality]. *But they are divided by different things, and you can't do that."*

Stella: "*I don't agree. I think it works for any number because I worked it out*

$$4x + 12 < 6x + 18$$
$$4x < 6x + 6$$
$$0 < 2x + 6$$

See, 6 is always greater than 0."

While sharing justifications, ask probing questions to make students' thinking visible, such as:

- *I noticed you compared terms from each side of the inequality,* 18*and* 12 *and* $6x$ *and* $4x$*. Can you say more about that? Why does it make sense to look at these pairs of terms?*
- *What do you mean when you say, "no matter what"?*
- *I noticed you recognized that they had the same expression,* $x + 3$*, on both sides of the inequality at one point. Why do you think this student took time to factor each expression? What might be helpful about this form?*

As discussion takes place, invite classmates to join the discussion. Ask:

- *What about this reasoning makes sense?*
- *Is there anything you disagree with?*
- *What questions surface for you?*
- *Are there some values of* x *where you are convinced this statement is absolutely true? What values might not make this inequality true?*
- *How might the factored form of the inequality help us confirm these ideas?*

Provide time for students to revise their work.

In this example, students are:

- given a cognitively demanding task. Students are asked to analyze a peer's response and decide if they agree or disagree. In doing so, students have an opportunity to construct a viable argument. Students have multiple entry points to reason about this prompt in various ways (LP 1).
- using the varied structures that give opportunities to determine an approach by working independently, then sharing with a partner or class, then revising their response based on those interactions (LP 2).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).

Example 2

Students are given a problem like this:

Which of the following equations have the same solution as $x + 3 = 5x - 4$*?*

A. $x - 3 = 5x + 4$

B. $2x + 8 = 5x - 3$

C. $10x + 6 = 2x - 8$

D. $10x - 8 = 2x + 6$

E. $0.3 + \frac{x}{10} = \frac{1}{2}x - 0.4$

Prepare to share your response.

- ➢ *Explain how you could tell which equations have the same solution as the given equation without solving each one (and without substituting a value).*
- ➢ *Are there other equations on the list that share a common solution? Without solving, how could you tell?*

(Adapted from Illustrative Mathematics, Same Solutions?, 2016)

Sample Learning Experience

After some individual work time where students develop an approach to the prompt, encourage students to share and compare approaches in pairs. Monitor and select a couple of approaches to highlight and reflect on progress made at this point. Look for student work that indicates students are beginning to attend to the structure of the equation.

Questions that can be posed to student pairs as they work:

- *What do you notice about the structure of the equations? In what ways are they the same? In what ways are they different?*
- *What relationships do you notice in the parts that look similar?*

Give selected pairs of students time to share responses and invite classmates to reflect on the following questions:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure and relationship between the equations to determine which sets of equations have the same solution. Give students additional time to revise their work.

In this example, students are:

- given a cognitively demanding task. They are asked to analyze the structure and relationships between equations to determine which equations have the same solutions. Students have multiple entry points to reason about these equations in various ways (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking process and revise their own thinking (LP 2).
- attending to the structure of the equations and working to make sense of the relationships between the equations. In so doing, students learn to recognize that different forms of an equation are helpful for different purposes (LP 3).

- encouraged to develop skills of expert learners, thereby developing agency, as they reflect on their approach to the prompt and incorporate new ways of thinking about this from their peers (LP 4).

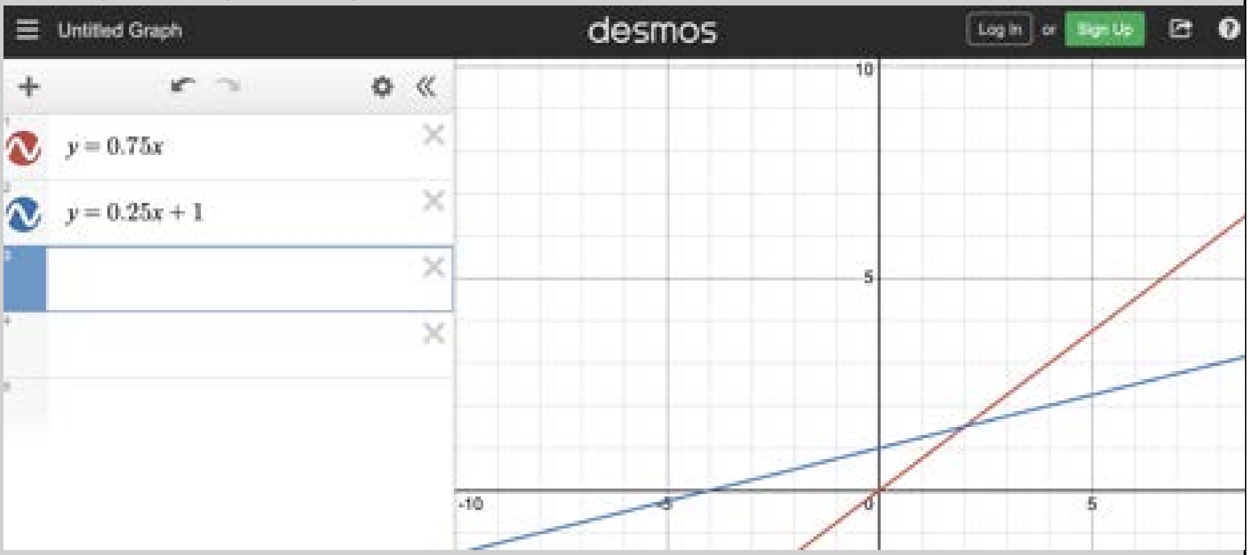
Example 3

Students are given an equation like this:

Consider the equation $0.75x = 0.25x + 1$

- ➢ *Think about the expressions "*$0.75x$*" and "*$0.25x + 1$*." Create a real-world situation that could be represented by these expressions.*
- ➢ *Based on this, what could x represent?*
- ➢ *Why would it make sense to have these two expressions be equal to one another based on the context?*
- ➢ *How could you use technology to create a graphical representation that shows the solution to the equation* $0.75x = 0.25x + 1$*?*

Sample graph using technology:



- ➢ *What is the solution to* $0.75x = 0.25x + 1$*? What does the solution to the equation mean relative to the context you created?*
- ➢ *What is another way you could have found the solution to* $0.75x = 0.25x + 1$*?*

Sample Learning Experience

The goal of this task is to invite students to make sense of an alternate way of thinking about solutions to an equation in order to build towards a flexible way of working with equations. In this case, students consider what each expression of an equation could represent in terms of a system of equations. This builds conceptual understanding (LP3).

In the first prompt, we are inviting students to give light context to the problem to help them personalize it and make concrete some of the conceptual ideas they are asked to explore.

Rephrase the prompt as necessary*: What quantity/situation could 0.75x represent? What quantity/situation could 0.25x + 1 represent?*

Consider selecting a context or two for reflection. Questions worth exploring are:

- *What about this context makes sense to have two expressions that are equal to each other?*
- *What about the context you developed makes sense? What, if anything, needs to be modified so that it makes sense?*

After this discussion, students should be given time to refine the context they developed.

Later, in the exploration, we want students to notice that this equation ($0.75x = 0.25x + 1$) is in terms of *x* and its solution is the *x*-coordinate of the point of intersection of the graphs of $y = 0.75x$ and $y = 0.25x + 1$. This conversation can be expanded to examine the nuanced difference between the solution to $0.75x = 0.25x + 1$ and the solution to the system of equations defined by $y = 0.75x$ and $y = 0.25x + 1$.

This work can be expanded by asking students to also represent equations with one solution, no solution, or many solutions graphically, as well as extend their exploration to include inequalities.

During the discussion of their work, ask students to reflect and share on these questions:

- *What does it mean to solve an equation? How about a real-life problem?*
- *What is helpful about considering the graph of the system of equations?*
- *How might you use what you learned through this experience in other settings?*

In this example, students are:

- given a cognitively demanding task. They are asked to consider an alternative way of thinking about an equation, develop a context that could be represented by the equation, and interpret the meaning of the solution (LP 1).
- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions of their peers to better understand the thinking process (LP 2).
- invited to give a context to the mathematical expressions to support their ability to communicate their conceptual understanding of the meaning of the solution to a mathematical equation instead of focusing on the procedural steps for solving. Students also work to reason about a specific equation in terms of a system of equations, making way for a deeper understanding of solving equations (LP 3).
- invited to give context to the mathematical expressions, giving students an opportunity to represent their interests and experiences in a mathematical way (LP 4).
- leveraging technology to examine a graphical representation of given expressions and to support their conceptual understanding of the quantities represented by the expressions, the meaning of the variable(s), constants, and the meaning of any solution(s) (LP 7).

Example 4

Students are given a problem like this:

Sandra works as a fitness manager at her local gym. She earns enough to pay for her living expenses and save $2000 per month to add to her savings account. She has already saved $12,000 for emergencies, which she estimates to be enough for 6 months of expenses.

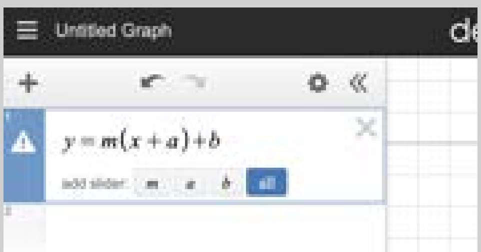
- **A.** *Write an equation to represent the balance in Sandra's account after $x$months.*
- **B.** *Graph this equation using Desmos.com or geogebra.org.*
- **C.** *Describe the key features of the graph. Then, explain them in the context of Sandra's savings.*

Next, use graphing software to explore changes to the equation and graph as Sandra encounters different scenarios. Use sliders by typing in "$y = m(x) + b$" and add "all" sliders on Desmos or refer to this video to see how to do it on Geogebra.

1. *What would change in the equation and graph if Sandra got a raise? Why does that make sense?*
2. *What would change in the equation and graph if Sandra changed the amount she saved every month? Why does that make sense?*
3. *What connections do you see between the equation you created in A above and the one you've been exploring?*

Extend the Learning:

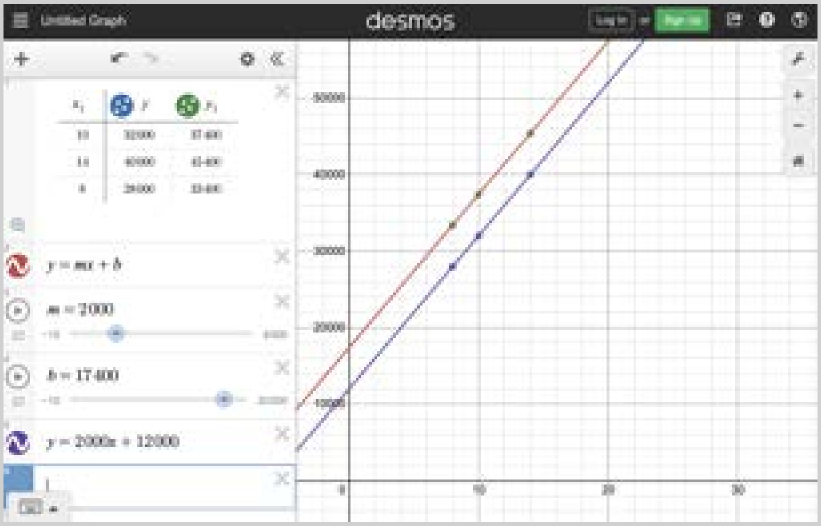
Next, let's build on this conversation and focus on the connection between equations and graphs. Select one of the following equations: $y = m(x + a) + b$ or $y = m(x - a) + b$. Then, add "all" sliders to your chosen equation (see image).



- *Use the sliders to explore the changes that occur in the graph when the values of m, a, and b are manipulated.*
- *How would you explain the impact that manipulating the value of a has on the graph of the linear equation?*

Sample Learning Experience

Give students an opportunity to independently begin exploration, then invite them to work in pairs to engage in the exploration. As students work, monitor for the ways they communicate their reasoning. Some students may use features of the graphing software to focus on particular points of interest, while others may create a table of values to compare and illustrate the changes they see.



Facilitate the discussion about the exploration.

Here are some points to emphasize:

- Highlight connections between the context, features of the graph (slope and y-intercept), and impact on the graph as constraints and assumptions change. For example, if Sandra gets a raise, the line has a greater slope and the line is steeper. The rate of change is also greater.
- Encourage students to compare the structure of the resulting equations, when parameters changed, with the structure of the original equation they determined in A.
- Invite students to explore and describe the impact of having this starting equation: "$y = m(x + a) + b$" or "$y = m(x - a) + b$".

Give students additional time to revise their work.

Additional Resources:

- 25 Jobs That Pay $50K a Year Without a Degree
- Jobs That Pay Over $50k
- 16 jobs that pay $40,000

In this example, students are:

- given a cognitively demanding task. They are asked to explore and explain how changes in parameters impact the graph of a linear equation, as well as how those changes connect to a real-life context (LP 1).
- working individually and collaboratively as they make progress through this exploration. Students are encouraged to ask questions to their peers to understand one another's thinking process (LP 2).
- given a task that centers reasoning with multiple opportunities to make sense of the impact of manipulating parameters in a linear equation on its graph. Students build in-depth understanding as they work to justify these changes (LP 3).

- given access to a graphing tool to aid in their understanding of and reasoning with multiple representations of the equation of a line (LP 7).

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M102 Modeling with Linear Functions and Equations

Badge Catalog Description

Which cell phone plan holds the best value for me? What combination of aluminum and iron will make the optimal container? Modeling with functions is a powerful way to gain insight into the world around us. Nature, society, business, and everyday life are full of situations in which a certain quantity depends on other measurable quantities. Function models enable us to describe, analyze, optimize, and predict what could happen in these situations. Meanwhile, equation models allow us to express real-world constraints in mathematical language. Solving the equations gives us new and useful information about the situation.

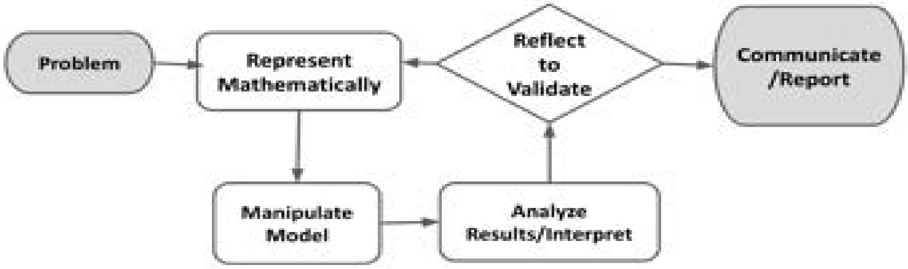
In M102 Modeling with Linear Functions and Equations, you will create function models in situations where changes in one quantity are assumed to be proportional to changes in another quantity. You will solve problems and interpret models to gain insight into situations involving a constant rate, such as the startup cost for a business, constant speed, uniform density, and many others. You will also create linear equations to model constraints in real-world problems, such as problems involving finite resources, spatial constraints, manufacturing specifications, or other conditions that must be satisfied. As you learn linear modeling, you will reason quantitatively using the relationships between the parts of a linear function model and the situation it describes. You will use technology as a tool to solve linear equations and to understand how the graphs of linear functions relate to their constituent parts. The work for this badge builds a foundation for the future study of topics such as quadratic and exponential function models. Modeling with linear functions and equations is useful for careers in a variety of fields, like science, engineering, medicine, business, and artificial intelligence.

Suggested prerequisites for this badge: comfort with solving simple equations and problems involving proportional relationships.

This badge is suggested as a prerequisite for: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

The M102 Modeling with Linear Functions and Equations badge integrates mathematical modeling as an essential component of how students engage with linear functions and equations. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M102 Modeling with Linear Functions and Equations center on the CCSSM modeling cycle described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, Modeling, K-12, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with linear functions and equations, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M102 badge.

M102 Content and Practice Expectations

102.a	Engage in the modeling cycle.

102.b	Interpret linear functions and equations that arise in applications in terms of the context.
102.c	Analyze linear functions using different representations.
102.d	Build linear functions that model relationships between two quantities.
102.e	Analyze and solve linear equations and pairs of simultaneous linear equations to draw conclusions.
102.f	Interpret expressions for linear functions in terms of the situation they model.
102.g	Solve linear equations and inequalities in one variable.
102.h	Create equations that describe linear relationships.
102.i	Understand the relevance of modeling with linear functions.
102.j	Use a linear function model to determine values of interest in a real-world problem.

Learning Principles

In M102 Modeling with Linear Functions and Equations, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges—for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and to communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has value in solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. Through reasoning and using multiple representations, students learn why procedures work and build a conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home

lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow students to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing the mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analysis. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M102

Whereas a typical instructional unit on linear functions might begin with a focus on students independently performing computations or manipulating linear expressions disconnected from real-world contexts, in M102, students should:

- regularly encounter real-world tasks involving two quantities whose measures vary such that the rate of change between the two is constant, and that require them to employ models and predict values. Common settings having constant rate of change include startup costs, constant speed, wages and salaries, manufacturing costs, fundraising, and rental charges (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially naming their own assumptions and variables and defending their choice of model as much as possible (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their solution methods (LP 2).

Spending time computing or manually completing tables with linear expressions should not be a focus of this badge. In fact, the opposite is true. In M102, students should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models as well as leverage tools to help solve problems. For example, students should:
    - explain the real-world meaning of a linear model.
    - explore and compare multiple different linear models.
    - explain the relationship between parameters in a linear model and features of its graph.
    - use a linear model to justify a claim about a real-world context.
    - justify the selection of a linear model over other models (LP 3).
- have frequent opportunities to use reasoning to relate the algebraic form of a linear function to its graph (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Instead of focusing only on the applications typically thought of as associated with linear functions, students should also be given opportunities to understand and reflect on the ways that linear modeling can authentically involve real-world situations (LP 6). Some contexts to consider include:

- personal income growth.
- cost of living impacts on business.
- analysis of pay differences.

Often, coursework with linear functions is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use spreadsheets and graphing software to allow for focus on understanding, rather than computation or symbolic manipulation. Students should write spreadsheet formulas to model phenomena that exhibit a linear model (LP 7).
- use motion detectors to measure distance vs. time (LP 7).

Evidence of Learning

In M102 Modeling with Linear Functions and Equations, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND

(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	**Indicators** **Choose an artifact where you...**
102.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
102.b: Interpret linear functions and equations that arise in applications in terms of the context.	i. select two different data points that illustrate something interesting or important about the context you are modeling and describe what these points tell you about the topic.
	ii. use a model to compute output values and then interpret the values in the context of the problem.
	iii. describe the domain of the linear function, including anything noticed about values that are not part of the domain given the context.
	iv. describe the contextual meaning of the rate of change for a linear model.
102.c: Analyze linear functions using different representations.	i. create a linear graph to show a particular context and describe important values that help understand something about the problem you are exploring.
	ii. explain how the parameters of a linear model relate to its graph.
102.d: Build linear functions that model relationships between two quantities.	i. identify the quantities and variables of interest in a given situation or data set that are interesting and that demonstrate a linear relationship.
	ii. build a function that models the linear relationship between two quantities in a situation.
102.e: Analyze and solve linear equations and pairs of simultaneous linear equations to draw conclusions.	i. find the solution to a system of two linear equations in two variables.
	ii. explain whether a system of two linear equations will have one solution, no solution, or an infinite number of solutions.
	iii. create a system of two linear equations in two variables to model a real-world situation.
	iv. explain the meaning of a solution to a system of two

Content and Practice Expectations	Indicators Choose an artifact where you...
	linear equations in two variables in terms of the context being modeled.
102.f: Interpret expressions for linear functions in terms of the situation they model.	i. describe the meaning of the parameters for a linear function in terms of the context.
102.g: Solve linear equations and inequalities in one variable.	i. find the solution to a linear equation or inequality in one variable.
	ii. justify a solution to a linear equation or inequality in one variable using logical reasoning or an argument about reasonableness.
102.h: Create equations that describe linear relationships.	i. identify the variables of interest in a given situation or data set that are interesting and that demonstrate a linear relationship.
	ii. build an equation that models the linear relationship between two quantities in a situation.
102.i: Understand the relevance of modeling with linear functions.	i. notice something that surprised you or that you learned about the topic through linear modeling of data. Share what surprised you or what you learned.
	ii. highlight something you are most proud of in your learning about linear modeling.
	iii. explain how modeling can be used to answer important questions.
102.j: Use a linear function model to determine values of interest in a real-world problem.	i. determine one or more output values for specific inputs to the function.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may	The student's artifact demonstrates evidence that they have met the expectations of the

indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	indicator(s).

Annotated Examples M102 Modeling with Linear Functions and Equations (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this to interpret the parameters of a linear function:

Part 1
Lauren records her taxi fare displayed on the computer screen at different points of time during a taxi ride:

Distance, *d*, in miles	**Fare, *F*, in dollars**
3	8.25
5	12.75
11	26.25

How would you describe the relationship between the two quantities?

Part 2
Identify two quantities that have a relationship that you'd like to track and believe is linear in nature.

- *What quantities are you tracking?*
- *What about the relationship between these two quantities leads you to believe it will be linear?*
- *Develop a model that describes the relationship between the two quantities.*
- *What surprised you about the relationship you just modeled?*

(Adapted from Illustrative Mathematics, Taxi!, 2016)

Sample Learning Experience

Set the stage by asking students, "*Why might Lauren keep track of the distances she travels and the amount she pays?*"

Ask students if they know of other situations where they might want to keep track of what they are doing and why. Note these ideas for the second prompt to extend the learning.

Monitor students as they work on their description of the relationship between the distance and the fare Lauren paid. Select student work that illustrates different techniques: use of a graph, analysis of rates of change within the table, or the writing of an equation. Consider sequencing the work in a way that builds a coherent mathematical story and allows for connections across representations. Encourage students to ask questions to their peers as mathematical ideas are presented.

Amplify aspects of the conversation that:

- explain the linear nature of this relationship.
- contextualize the slope and y-intercept of the line.
- lead to the equation of this linear function.

Allow students the opportunity to return to their work to revise, as necessary.

Have students work in pairs to complete the task. Allow them the option of using chart paper, Google Slides, or other creative outlets to showcase their work.

In this example, students are:

- given a conceptual task reflecting a real-world scenario that requires students to convey their understanding. They are also asked to consider improvements to their work (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction among peers (LP 2).
- applying their understanding of linear functions to identify and model a relationship between two quantities they believe to be linear in nature (LP 3).
- bringing their identities and interests to the fore as they identify a relationship between quantities they wish to explore (LP 4).
- using technology as a tool (LP 7).

Example 2

Students are given a problem like this:

While on an airplane, Leticia noticed that she could toggle back and forth between imperial and metric measurements showing data at a given time point during the flight. She took photographs of the imperial and metric data on two different flights.

Imperial and Metric Data from Flight 1

Destination METRIC Atlanta (ATL)	**IMPERIAL**
Time to Destination: 1:01	Distance to Destination: 406 mi
Local Time at Origin: 7:41 AM	Local Time at Destination: 7:41 AM
Altitude: 28 ft	Head Wind: 0 mph
Outside Temperature: 66°F	Ground Speed: 11 mph

Destination **METRIC** Atlanta (ATL)	IMPERIAL
Time to Destination: 1:01	Distance to Destination: 653 km
Local Time at Origin: 07:41	Local Time at Destination: 07:41
Altitude: 9 m	Tail Wind: 0 km/h
Outside Temperature: 19°C	Ground Speed: 17 km/h

Imperial and Metric Data from Flight 2

Destination METRIC Atlanta (ATL)	**IMPERIAL**
Time to Destination: 3:23	Distance to Destination: 1591 mi
Local Time at Origin: 3:40 PM	Local Time at Destination: 5:40 PM

Altitude: 4279 ft	Head Wind: 19 mph
Outside Temperature: 77°F	Ground Speed: 0 mph

Destination IMPERIAL **METRIC** Atlanta (ATL)	
Time to Destination: 3:23	Distance to Destination: 2561 km
Local Time at Origin: 15:40	Local Time at Destination: 17:40
Altitude: 1304 m	Head Wind: 30 km/h
Outside Temperature: 25°C	Ground Speed: 0 km/h

Sample Learning Experience

Share images with students and ask, *"What do you notice? What do you wonder?"*

Generate a list of what students noticed and wondered and amplify their wonderings that lead to an exploration of linear functions, such as the relationship between imperial and metric measurements of temperature, altitude, or distance to destination.

In groups, have students select a measure to focus on, like temperature, and consider how they can model the relationship between imperial and metric methods of measurement.

Allow groups time to consider the question and identify possible steps to approach the question. Encourage students to use graphing technology as they complete the task.

Bring the groups together to allow them to share ideas on how they are approaching the task. Use the following questions to discuss highlighted ideas:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Follow this intermediate discussion with the following prompt for groups to complete. Groups should reflect on and revise their thinking based on the ideas shared during the discussion.

1. *Write an equation that takes a measure from metric to imperial.*
2. *If you were to make the graph of the equation you wrote, how well does it fit the data you have? What might account for any differences you see?*
3. *For someone who starts with a measure in imperial, how might they use your equation to get the metric equivalent?*
4. *What additional questions could be answered about this context using some of the models we've seen? Are some models more helpful than others in answering these additional questions?*

Circulate among the groups and take note of various solution paths they are using to approach the problem. Identify groups that will present to the whole class. Be mindful to emphasize different approaches to the problem. Consider a sequence of presentations to build a mathematical story. Encourage students to ask questions to their peers as solutions are presented.

Possible points to emphasize during a discussion of responses to these questions:

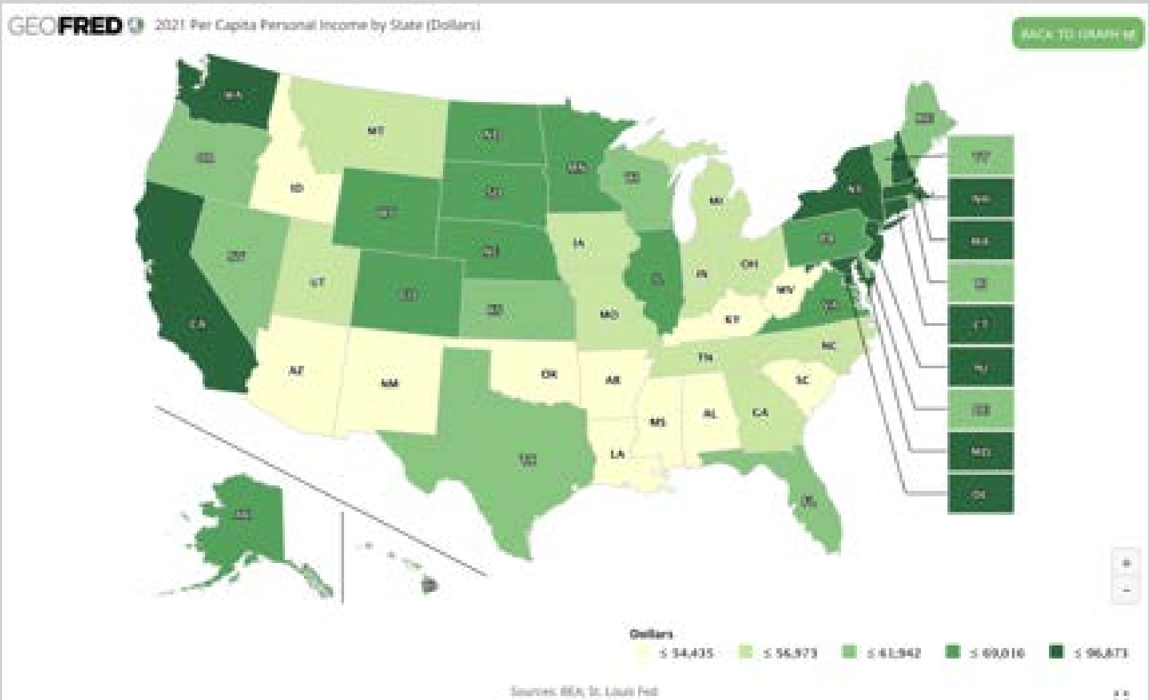
- For question 1, look for students who organize information in a table or graph to then determine an equation. Focus discussion on eliciting student reasoning.
- For question 2, listen for students who attribute discrepancies to measurement or rounding error. The airplane's instruments may only display the whole number value.
- For question 3, engage students in repeated reasoning that leads to an equation for the inverse. Name the relationship between the two equations, metric → imperial and imperial → metric, as inverse from the work they produce. Ask students to reflect on the usefulness of each form.
- Push student thinking by asking them to consider the following questions:
    - *Is there a point that lies on both lines simultaneously if both equations were graphed on the same coordinate plane?*
    - *What is the significance of this point?*

In this example, students are:

- given a cognitively demanding task. It requires that they analyze a situation and determine an approach to convert between two quantities without originally specifying a particular method, which means groups can create tables and graphs, and make predictions or create equations. The exercise begins with a broadly accessible task and questions—"What do you notice? What do you wonder?"—that support learners with varied prior learning experiences. Students are then guided to create a linear model and interpret that model. They are also asked to consider improvements to their model (LP 1).
- working collaboratively in groups to complete the task, allowing for social interaction among peers. Students are also encouraged to ask questions to their peers to better understand the thinking process (LP 2).
- applying their understanding of linear functions to identify and model a relationship between two quantities (LP 3).
- connecting mathematics to situations they may encounter, thereby building a lens for using math to explain and make sense of relationships between quantities in their lives (LP 4).

Example 3

Have student pairs complete a task like the following:



(FRED Economic Research, 2022)

Explore how the per capita personal income (PCPI) in your state has grown in the last 10 years. Access options here: *Per Capita Personal Income by State, Annual.*

- ***A.*** *Extract enough data to determine a linear model that describes how the PCPI measure has grown over time in your state.*
- ***B.*** *Interpret each number in the linear equation, identifying what each means in the problem's context.*
- ***C.*** *Use your linear relationship to predict PCPI in 2025.*
- ***D.*** *How useful is a linear model in describing your state's PCPI over time? Why do you think so?*
- ***E.*** *Explore the PCPI data for two states of your choice using the data from the Regional Economic Analysis Project. What do you notice? What do you wonder? What do you think accounts for the differences you see? How do the data highlight potential inequities? Prepare a presentation on your findings to share with others.*

Sample Learning Experience

Share the image of 2021 Per Capita Personal Income by State (dollars) and ask students, "*What do you notice? What do you wonder?*" [Alternatively, you can orchestrate a Slow Reveal of the graph.] Make a note of student wonderings as this may serve as the basis for further investigation.

Consider giving time for students to explore the significance of PCPI in order to optimize the chance for reflection on the usefulness of their model in prompt D. Ask students to reinterpret what these numbers represent within their local context.

As students progress through the prompts A-D, pause at strategic places to reflect on how they are approaching this work. Invite selected students or working pairs to share their reasoning and approach. Give space for students to adapt and revise their own work.

Consider using the following prompts:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Invite student pairs to showcase their work on prompt E. They could use chart paper, presentation software, or apps (Google Slides, PowerPoint, etc.).

Additional Resources:

- Why current definitions of family income are misleading, and why this matters for measures of inequality
- Per Capita Income Data Table
- Per Capita Income Explanation

In this example, students are:

- given a conceptual task reflecting a real-world scenario that requires them to use their knowledge of linear functions to model relationships between quantities and convey their understanding of key contextual and mathematical features. The exercise begins with a broadly accessible task and questions—"What do you notice? What do you wonder?"—that support learners with varied prior learning experiences (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction among students, as well as to gain insights about the significance of PCPI data from each other (LP 2).
- bringing their identities and interests to the fore as they identify the states they would like to explore and conduct an analysis (LP 4).
- applying mathematics to build an effective argument. Students will see how the Per Capita Personal Income varies throughout the US and how it indicates an area's wealth or scarcity, thus allowing for important policy decisions (LP 6).
- using technology as a tool to check their prediction (LP 7).

Example 4

Share a table that outlines median monthly rent by state, like the one below.

Ask: *"What do you notice? What do you wonder?"*

State	Annual Mean Wage (All Occupations)	Median Monthly Rent	Value of a Dollar
Alabama	$46,840	$998	$1.16
Alaska	$61,760	$1,748	$0.95
Arizona	$53,400	$1,356	$1.04
Arkansas	$44,780	$953	$1.17
California	$65,740	$2,518	$0.87
Colorado	$60,840	$1,927	$0.98
Connecticut	$65,450	$1,803	$0.94
Delaware	$56,700	$1,435	$1.01
Florida	$50,020	$1,590	$0.99
Georgia	$51,940	$1,262	$1.08
Hawaii	$58,190	$2,481	$0.85

(Source: The Average Cost of Living by State and Why Ignoring It Could Sink Your Business)

Your job is to explore how different income structures impact saving for an emergency fund. Consider the following situation.

Paula works as an electrician and makes about $52,000 a year. Her living expenses are about $2,000 a month. She has saved $12,000, which she estimates is enough for 6 months of expenses in case of emergency.

- **A.** *Write an equation that represents how much Paula has after x months.*
- **B.** *Graph this equation using Desmos.com or geogebra.org.*
- **C.** *Describe what the graph looks like.*
- **D.** *Why does it make sense that the graph looks like that?*

(Source: 25 Jobs That Pay $50K a Year Without a Degree)

Irfan works as a machine assembler for $17 an hour and has about the same cost of living as Paula. He works full-time, which is 40 hours a week. He would like to save for 6 months of expenses.

- **A.** *Write an equation that represents how much Irfan has after x months.*
- **B.** *Graph this equation using Desmos.com or geogebra.org.*
- **C.** *Describe what the graph looks like.*
- **D.** *Why does it make sense that the graph looks like that?*

(Source: Find Jobs: Assembler)
(Additional Resource: Beyond the Farm: Rural Industry Workers in America)

Consider Paula and Irfan's income structures, living expenses, and their wish to have emergency fund savings. Use features of your graphing software to explore the following:

- *Under what conditions will Irfan be able to save $12,000?*
- *Explore the legal requirements for working overtime. Resource: Overtime Pay*
- *How many additional hours a week (overtime) will Irfan need to work if he gets paid at a rate of time and a half? Or double his regular rate?*
- *Prepare to share about the differences you notice between the two workers.*

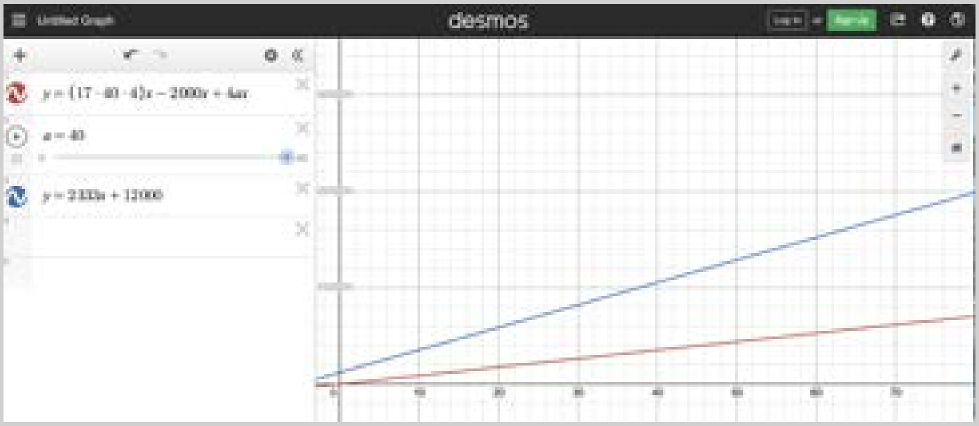
Next, explore income structures for jobs in a field of interest to you. One possible resource is May 2021 National Occupational Employment and Wage Estimates: United States

Develop a plan for saving for 6 months of expenses for an emergency.
- *What are the typical living expenses for your area?*
- *What is an equation that models how much money you'd have after x months?*

Sample Learning Experience

Begin by giving students an opportunity to notice and wonder about data found on a median monthly rent table. Build on their wonderings to introduce the idea of monthly living expenses. Ask students to identify a state where $2,000 would be a reasonable estimate of monthly living expenses.

Give students time to begin exploration independently, then invite them to work in pairs to engage in the exploration. As students work, monitor for ways they communicate their reasoning.
Some students may use features of the graphing software to focus on particular points of interest, while others may create a table of values to compare and illustrate the changes they see.



Facilitate the discussion about the exploration. Here are some points to emphasize:

- Connections between the context, features of the graph (slope and y-intercept), and impact on the graph as constraints and assumptions change. For example, if Paula gets a raise, the line has a greater slope and the line is steeper. The rate of change is also greater.
- Encourage students to compare the structure of the resulting equations, when parameters changed, with the structure of the original equation they determined in A.

Give students additional time to revise their work.

In this example, students are:

- given a cognitively demanding task. The exercise begins with a broadly accessible task and questions—"What do you notice? What do you wonder?"—that support learners with varied prior learning experiences. Students are asked to explore and explain how changes in parameters impact the graph of a linear equation, as well as how those changes connect to a real-life context (LP 1).
- working individually and collaboratively as they make progress through this exploration. Students are encouraged to ask questions to their peers to understand one another's thinking process (LP 2).
- given a task that centers reasoning with multiple opportunities to make sense of the impact of manipulating parameters in a linear equation on its graph. Students build in-depth understanding as they work to justify these changes (LP 3).
- given access to a graphing tool to aid in their understanding of and reasoning with multiple representations of the equation of a line (LP 7).

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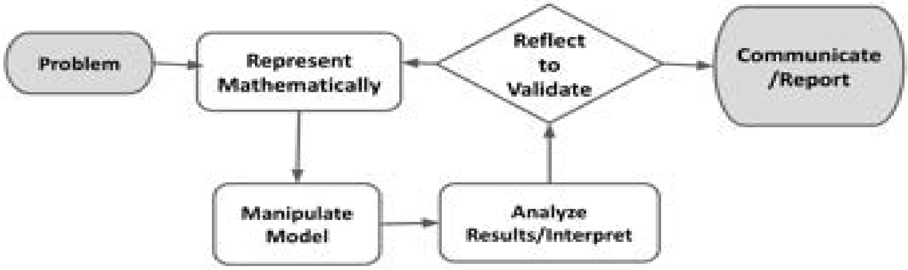
M103 Modeling with Functions of Quadratic Type

Badge Catalog Description How can we predict the path of a rocket through the sky? What quantity of sales will maximize business profits? Modeling with functions is a powerful way to gain insight into the world around us. Nature, society, business, and everyday life are full of situations in which a certain quantity of interest depends on other measurable quantities. With function models, we can describe and analyze these situations and even optimize them and predict what will happen. In M103 Modeling with Functions of Quadratic Type, you will create and make sense of models in situations that suggest a quadratic relationship. You will solve problems and gain insight into situations like accelerated motion or determining how to produce the best outcome in a model that represents a real-world problem in the workplace, the environment, or society. As you learn about quadratic modeling, you will explore the relationships between the algebraic form of a quadratic function model and the situation it describes. You will use technology as a tool to solve quadratic equations and to understand how the graphs of quadratic functions relate to their algebraic form. The work for this badge builds a foundation for the future study of topics such as polynomials and complex numbers. Modeling with quadratic functions is useful for careers in a variety of fields, like engineering, military, law enforcement, astronomy, car manufacturing, and agriculture. Suggested prerequisites for this badge: M102 Modeling with Linear Functions and Equations. This badge is suggested as a prerequisite for: M201 Function Concepts; M202 Rational Exponents and Complex Numbers; M203 Polynomial and Rational Expressions, Functions, and Equations; M204 Exponential and Logarithmic Functions and Equations.

The M103 Modeling with Functions of Quadratic Type badge integrates mathematical modeling as an essential component of how students engage with functions of the quadratic type. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M103 Modeling with Functions of Quadratic Type center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, Modeling, K-12, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with quadratic functions, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M103 badge:

M103 Content and Practice Expectations

103.a	Engage in the modeling cycle.
103.b	Interpret quadratic functions that arise in applications in terms of the context.
103.c	Analyze quadratic functions using different representations.
103.d	Build quadratic functions that model relationships between two quantities.

103.e	Construct and compare linear and quadratic models, solve problems, and draw conclusions.
103.f	Interpret expressions for quadratic functions in terms of the situation they model.
103.g	Summarize, represent, and interpret data on two quantitative variables for linear and quadratic model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes informal description of the fit of the curve and addresses the usefulness of the model for the particular context.[1]
103.h	Understand the relevance of modeling with quadratic functions.
103.i	Use a quadratic function model to determine values of interest in a real-world problem.

Learning Principles

In M103 Modeling with Functions of Quadratic Type, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

[1] More formal statistical regression analyses are part of Badge M112.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M103

Often, the study of quadratic functions is drawn together with solving quadratic equations, factoring quadratic expressions, and even polynomial operations more generally. The result is often an unfocused experience that prioritizes factoring quadratics with integer coefficients. Most important, not enough emphasis is paid to the power of quadratic functions as modeling tools.

The focus of M103 reverses this trend, focusing student understanding on using quadratic functions for modeling purposes. Key characteristics of M103 include the following:

- A distinct shift towards using functions, represented algebraically, in tables, and in graphs, to examine real-world situations, including projectile motion, business profits and losses, design problems involving area, and more (LP 1).
- Real-world tasks should be at the center of all work with quadratic functions in this badge. Students should be given tasks that require engagement with multiple parts of the modeling cycle and should name their own assumptions and variables as much as possible (LP 1).
- Students should also be given choice of representation (table, equation, graph, or others) in creating models (LP 4).
- Students can engage socially by collaborating with each other and sharing and discussing their solution methods (LP 2).

- Tasks can be organized around topics, allowing students to have agency by choosing what they are interested in (LP 4).

Sometimes, in the study of quadratics, significant time can be spent computing with functions or graphing by hand. To aid understanding and allow for students to maximally engage in modeling, students should have access to computer graphing tools.

Students should also be given frequent opportunities to use spreadsheets to model quadratic phenomena, especially as a tool to understand first and second differences (LP 7). The use of technology allows for a focus on model creation and conceptual understanding, as opposed to computation.

Further, students should engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models. As examples, students should:

- explain the real-world meaning of a quadratic model.
- compare two different quadratic models.
- explain the relationship between parameters in a quadratic model and features of its graph.
- use a quadratic model to justify a claim about a real-world context.
- justify the selection of a quadratic model, based either on characteristics of the quantities (e.g., projectile motion) or analysis of the rate of change (e.g., that it is changing at a constant rate) (LP 3).

Many of these reasoning-based activities can be done with partners or groups of any size, allowing for social interaction as part of the learning process (LP 2). The history of mathematics also presents fertile ground for the study of quadratic functions (LP 5).

Instead of focusing only on the applications typically thought of as associated with quadratic functions (e.g., projectile motion), students should also be given opportunities to understand and reflect on the ways that quadratic modeling can use mathematics in authentic situations (LP 6). For example, students can use quadratic functions to examine innovations in recyclables and profits analysis.

Evidence of Learning

In M103 Modeling with Functions of Quadratic Type, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one performance assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
103.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
103.b: Interpret quadratic functions that arise in applications in terms of the context.	i. select two different data points that illustrate something interesting or important about the context you are modeling and describe what these points tell you about the topic.
	ii. use a model to compute output values and then interpret the values in the problem's context.
	iii. describe the domain of the quadratic function, including anything noticed about values that are not part of the domain or why the shape of the model might change for particular values of the domain.
	iv. describe the contextual meaning of the average rate of change between two points for a quadratic function.
103.c: Analyze quadratic functions using different representations.	i. create a quadratic graph to show a particular context and describe important values that help understand something about the problem you are exploring.
	ii. explain how the parameters of a quadratic model relate to its graph.
103.d: Build quadratic functions that model relationships between two quantities.	i. identify the variables of interest in a given situation or data set that are interesting and that demonstrate a quadratic relationship.
	ii. build a function that models the quadratic relationship between two quantities in a situation.
103.e: Construct and compare linear and quadratic models and solve problems.	i. distinguish between situations that can be modeled with linear functions and those that can be modeled with quadratic functions, based on the quantities involved or through analysis of the rate of change.

Content and Practice Expectations	Indicators Choose an artifact where you...
	ii. provide insight into why some relationships are linear and others are quadratic, or explain why, in terms of a situation being modeled, a quadratic function is a better model than a linear function.
103.f: Interpret expressions for quadratic functions in terms of the situation they model.	i. describe the meaning of the parameters for a quadratic function in terms of the context.
103.g: Summarize, represent, and interpret data on two quantitative variables for linear and quadratic model fits.	i. choose a particular representation of your data to communicate something important to a particular person or audience and explain why you made that choice. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes informal description of the fit of the curve and addresses the usefulness of the model for the particular context.
103.h: Understand the relevance of modeling with quadratic functions.	i. recognize something that surprised you or that you learned about the topic through quadratic modeling of data. Share what surprised you or what you learned.
	ii. highlight something you are most proud of in your learning about quadratic modeling.
	iii. explain how modeling can be used to answer important questions.
103.i: Use a quadratic function model to determine values of interest in a real-world problem.	i. determine one or more output values for specific inputs to the function.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

as evidence of the indicator(s).	corrected in revision. The student may revise the selected artifact or submit a different artifact.	

Annotated Examples M103 Modeling with Functions of Quadratic Type (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this to work on in groups:

Work with your group to develop a model that describes the height of a ball thrown at a given time during the throw. Have one person in the group throw a ball while another records a video of the event. Make sure that the video includes the entire path of the ball from the throw to when it hits the ground. Prepare your findings to share with the class.

Sample Learning Experience

Students could be given the largely unstructured task above, along with the tools needed to carry it out: a ball, recording technology, and tools to make mathematical models, like spreadsheets and access to graphing software. Consider bringing the class together at strategic points to facilitate progress toward the goal of developing a mathematical model.

In these instances, monitor, select, and sequence groups to report on their efforts.

The following questions can be used to facilitate discussion early in the process:

1. Watch a selected group's video. Ask students to describe what they notice while watching the video. Listen for descriptions of key elements, such as the height of the ball upon release, the highest point the ball reached, and when and where the ball landed.
2. Ask students, "*How might we represent this mathematically?*"
   As students share, make note of ideas that surface. When they use hand signals or share descriptions that allude to the graph of a quadratic function, encourage students to sketch what that might look like.

3. Once a sketch of a graph is offered, ask students, "*What quantities are captured here?*" in order to develop insights into what quantities they will work on to define and then gather data.
4. Use the list of what students noticed in #1 above to frame the next phase of investigation. For example, you may build from a statement like "The ball hits a high point before coming back down" to ask, "*How high was the ball above the ground at its highest point?*" Invite students to discuss and brainstorm the following question: "*What makes finding this difficult and how can you persevere?*"
5. Ask students to reflect on the question "*What can you take from this discussion to apply to your own work?*" before continuing with their investigation.

When ready, give time for groups to share their models with the class. Here are some questions to facilitate this discussion:

- *What do you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, students are given an opportunity to reflect on and revise their work.

In this example, students are:

- given a cognitively demanding task. It requires students to model projectile motion using a quadratic function. They must make many assumptions to design, investigate, and define their own variables. The exercise begins with a broadly accessible task and prompts—"Ask students to describe what they notice"—that support learners with varied prior learning experiences (LP 1).
- set up to have social interactions, as they work in small groups and discuss their work with each other. These interactions may happen live in a physical classroom or virtually (LP 2).
- deepening their understanding of functions used to model data. Students also have the opportunity to make connections between key features of the context, graph, and equation. As they consider the relationship between the quantities, students may recognize that a linear model has limited usefulness and therefore might prefer a nonlinear model, quadratic in nature (LP 3).
- able to explore skills and dispositions around what it means to model with mathematics. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to make sense of the world (LP 4).
- using technology as a tool, as they employ spreadsheet and/or graphing software to perform their analysis (LP 7).

Example 2

Students are given a problem like this to work on in groups:

Part 1: Making sense of rocket launches

1. *Watch this video of Grasshopper 744m Test| Single Camera (Hexacopter).*
2. *Reflect: What questions does this clip bring to mind?*
3. *Sketch a diagram that illustrates the rocket's flight. Use the diagram you developed to sketch a graph.*
4. *Read the article "Why SpaceX wants to land a rocket on a platform in the ocean" by Joseph Stromberg.*
5. *What questions do the article and the video answer? What new questions does this surface for you?*

Part 2: Modeling with Quadratics

After some trials, SpaceX learned to control the flight of reusable rockets used to launch spaceships into space.

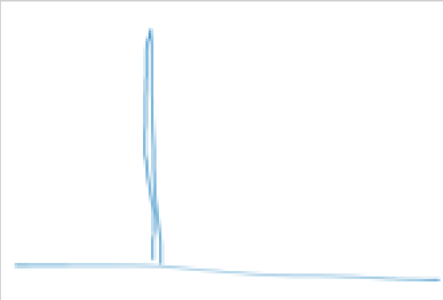
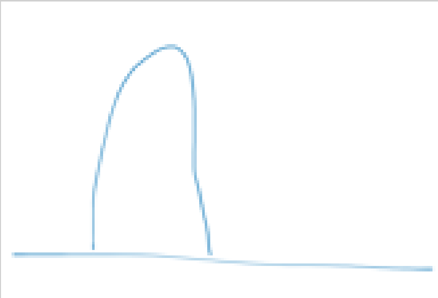
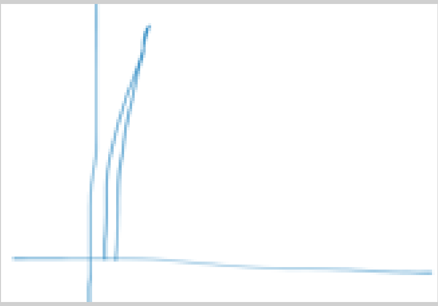
1. *Watch and gather data from SpaceX' April 2022 launch of Ax-1 [watch from about 3:26 to 3:36] to create a model for the flight of the Phase 1 rocket used in that mission.*
2. *Prepare to share your findings.*
3. *What new questions surface for you?*

Sample Learning Experience

Part 1: Making sense of rocket launches

Begin by inviting students to share what they know about space exploration. Listen for stories where students express curiosity about launching objects into space: how are things launched, do these objects ever come back, and what happens to objects once they are launched into space? Use these themes to set up SpaceX's work with reusable rockets. Share the video of Grasshopper's 744m Test (1:36) from their early efforts in the development of these rockets.

Monitor students as they work to sketch the rocket's flight. Select student responses to highlight. Some possible approaches may take the following forms:

As students share their sketches with classmates, invite them to make sense of each other's sketches. Consider the following questions:

- *What are some things you appreciate about this sketch?*
- *What questions does this sketch surface for you?*
- *Do you think the sketch is reasonable? Why or why not?*
- *What do your horizontal and vertical axes represent?*
- *What can you take from this approach to apply to your own work?*
- *What are some quantities we could track and work to represent in mathematical ways? What questions could we answer with that information?*

As you bring the discussion of Part 1 to a close, give students time to make note of new insights they have gained into rocket flights. Consider amplifying ideas around what quantities they want to track and ask how they might measure or get that information.

Part 2: Modeling with Quadratics

Share with students, "Since SpaceX's efforts in 2013 with the Grasshopper rocket, they have made tremendous progress."

Use thoughts students shared in discussion of Part 1 to set the stage for viewing the video of the April 8, 2022, Ax-1 Mission. This video tracks the time, speed (km/hr.), and altitude for both the Phase 1 and Phase 2 rockets. Encourage students to focus on the Phase 1 rocket.

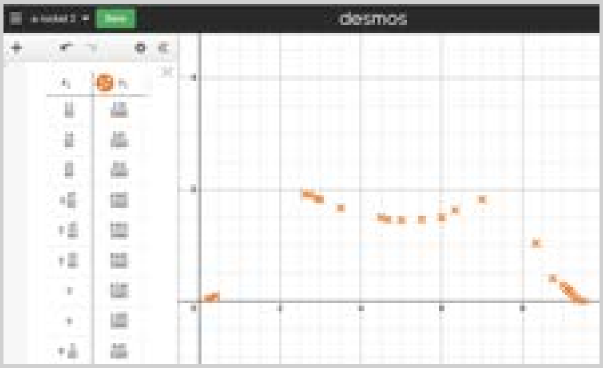
Give students a few minutes of individual work time before encouraging them to work in a small group. Students should have access to spreadsheets and graphing software or calculators. As they join their small groups, ask students to take time to share and make sense of each other's approach. As students share progress made within their small groups, select a couple of strategies to share with the whole class. The focus of this whole group discussion is to provide space for students to make sense of various approaches and gain insights into their own thinking. The goal is not to discuss a model just yet.

Consider using the following prompts to facilitate the conversation:

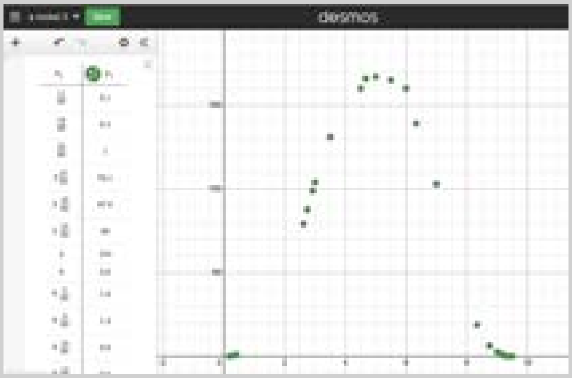
- *What makes sense about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the approach (or model) is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Some possible graphs students may develop:

Speed (km/s) vs. time



Altitude (m) vs. time



Give students additional time to use insights gained from sharing to revise and further develop their models. The image below illustrates a manual fit (purple) quadratic and one from a regression analysis. As student groups work, monitor and note different approaches they take to find a quadratic model they feel best fits the data gathered. Select varied approaches to share with the whole class.

Consider using the following prompts to facilitate the conversation:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Amplify aspects of the conversation that:

- attempt to explain the quadratic nature of this relationship.
- contextualize the key features of the graph.
- lead to the equation of this quadratic function.

Allow students the opportunity to return to their work to revise, as necessary.

Have students work in pairs to complete the task. Allow them the option of using chart paper, Google Slides, or other creative outlets to showcase their work.

In this example, students are:

- given a cognitively demanding task. It requires students to make sense of rocket trajectories and recognize which quantities can be modeled with quadratic functions. The task offers entry for students with a variety of prior learning experiences by beginning with student-posed questions (e.g., "What questions does this clip bring to mind?") and informal sketching of rocket trajectories (LP 1).

- set up to have social interactions, as they work in small groups and discuss their work with each other. These interactions may happen live in a physical classroom or virtually (LP 2).
- deepening their understanding of functions used to model data. Students also have the opportunity to make connections between key features of the context, graph, and equation (LP 3).
- able to explore skills and dispositions around what it means to model with mathematics. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to make sense of the world (LP 4).
- given insight into how companies work to develop an environmentally responsible approach to space exploration (LP 6).
- using technology as a tool, as they employ spreadsheet and/or graphing software to perform their analysis (LP 7).

Example 3

Students are given this problem like this to work on in groups:

The appearance of a rainbow is a phenomenon that has captured the imaginations of people since prehistoric times. At one point, the Greek philosopher Aristotle described the rainbow in Book III of his treatise Meteorology: *"The rainbow never forms a full circle, nor any segment greater than a semicircle. At sunset and sunrise the circle is smallest and the segment largest: as the sun rises higher the circle is larger and the segment smaller" (as quoted in Corradi's* A short history of the rainbow).

Notice that Aristotle proclaims, "the rainbow never forms a full circle." In this assignment, you will explore this phenomenon and use tools, including Desmos.com or Geogebra.org, to construct an argument that either supports or refutes Aristotle's statement.

Some materials to explore:

- *Kamal Al-Din Al-Farisi's Explanation of the Rainbow*
- *Corradi history of the rainbow.pdf*
- *Chasing Rainbows in the Hudson Valley*

Consider using sliders on Desmos or Geogebra to explore the graph of a quadratic function and the graph of a circle.

One form of a quadratic function you can use is $y = a(x - h)^2 + k$. *One form for the equation of a circle you can use is* $(x - d)^2 + (y - f)^2 = r^2$.

To fit these curves to an image of a rainbow, begin by adding an image of a rainbow onto your Desmos or Geogebra graph.

Prepare to share your findings in a presentation, including:

- *New understandings you have about rainbows from your research.*
- *Highlights from your exploration in fitting the graph of a quadratic function and fitting the graph of the equation of a circle to the image you used.*

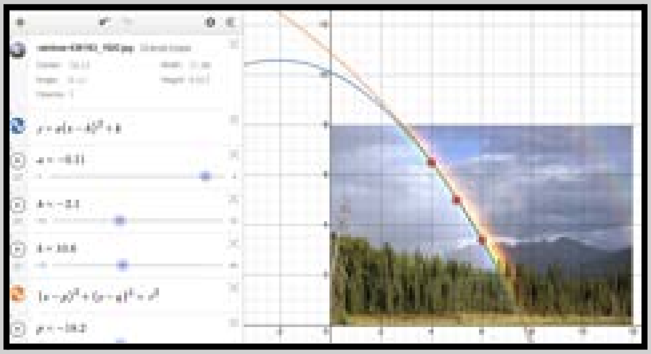
- *How you decided which was a better fit.*
- *The equation of the curve of best fit and your best explanation as to what each parameter could possibly represent.*

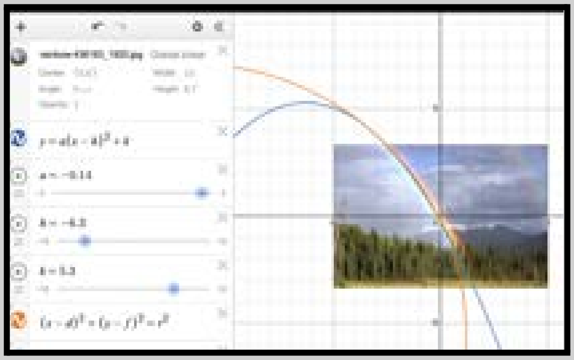
Here is one sample image you may choose to use.

(Source: Pixabay picture: Rainbow Bow Colors Double - Free photo on Pixabay)

Sample Learning Experience
Give students an opportunity to begin exploration independently, then invite them to work in pairs to engage in the exploration. As students work, monitor for ways they communicate their reasoning.

Some students may use features of the graphing software to focus on particular points of interest, while others may create a table of values to compare and illustrate changes they see.





Facilitate the discussion about the exploration.

Here are some questions to emphasize:

- *How do they choose to place their image on the grid?*
- *How did they verify the focus of their work: did they select key points, color, or some other feature(s) to guide them as they explored best fit?*
- *How did they determine which was a better fit?*
- *How did their research influence their approach?*

Additional questions for reflection:
- *What's promising about this approach?*

- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Give students additional time to revise their work.

In this example, students are:

- given a cognitively demanding task. It requires students to determine how to best develop an argument as to which graph is a better model for the curve of a rainbow. The task offers entry for students with a variety of prior learning experiences by beginning with the use of an image of a rainbow, various tools, and resources to explore options. Through iterative cycles of work, discussion, and research, students have the opportunity to improve on preliminary models and construct a viable argument (LP 1).
- set up to have social interactions, as they work in small groups and discuss their work with each other. These interactions may happen live in a physical classroom or virtually (LP 2).
- given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening (LP 3).
- able to explore skills and dispositions around what it means to model with mathematics. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to make sense of the world (LP 4).
- able to view mathematics as a human endeavor across centuries. In exploring resources students gain insights into the ideas that shaped current understanding of rainbows. They have the opportunity to see the iterative nature of the development of scientific and mathematical ideas spanning centuries (LP 5).
- using technology as a tool, as they employ spreadsheet and/or graphing software to perform their analysis (LP 7).

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M104 Modeling with Functions of Exponential Type

Badge Catalog Description

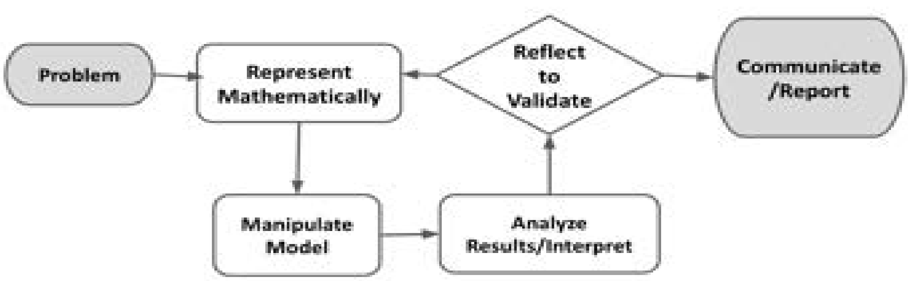
What will the money in your savings account be worth in 30 years? How does a coroner determine the time of death? Nature, society, business, and everyday life are full of situations in which a certain quantity depends on other measurable quantities. Function models enable us to describe, analyze, optimize, and predict what can happen in these situations. In M104 Modeling with Functions of Exponential Type, you will create function models in situations where a quantity changes at a rate that is proportional to its value. You will solve problems and reason about situations of exponential growth and decay, such as wildlife populations, financial investments and depreciation, bacterial growth, radioactivity, internet usage, or the popularity of fads. As you learn exponential modeling, you will make sense of the relationships between the algebraic form of an exponential function model and the situation it describes. You will use technology as a tool to solve exponential equations and to understand how the graphs of exponential functions relate to their algebraic form. The work for this badge builds a foundation for the future study of topics such as rational exponents and logarithmic functions. Modeling with exponential functions is useful for careers in a variety of fields, like economics, biology, sound engineering, and statistics.

Suggested prerequisites for this badge: M102 Modeling with Linear Functions and Equations.

This badge is suggested as a prerequisite for: M201 Function Concepts; M202 Rational Exponents and Complex Numbers; M203 Polynomial and Rational Expressions, Functions, and Equations; M204 Exponential and Logarithmic Functions and Equations.

The M104 Modeling with Functions of Exponential Type badge integrates mathematical modeling as an essential component of how students engage with functions of the exponential type. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M104 Modeling with Functions of Exponential Type center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards. (Adapted from Common Core Standards Writing Team, 2019, Modeling,K-12, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with exponential functions, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M104 badge.

M104 Content and Practice Expectations

104.a	Engage in the modeling cycle.
104.b	Interpret exponential functions that arise in applications in terms of the context.
104.c	Analyze exponential functions using different representations.
104.d	Build an exponential function that models a relationship between two quantities.
104.e	Construct and compare linear and exponential models and solve problems to draw conclusions.
104.f	Interpret expressions for exponential functions in terms of the situation they model.

104.g	Summarize, represent, and interpret data on two quantitative variables for linear and exponential model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes the informal description of the fit of the curve and addresses the usefulness of the model for the particular context.[2]
104.h	Understand the relevance of modeling with exponential functions.
104.i	Use a functions of exponential type model to determine values of interest in a real-world problem.

Learning Principles

In M104 Modeling with Functions of Exponential Type, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as

[2] More formal statistical regression analyses are part of Badge M112.

mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M104

Whereas a typical instructional unit on exponential functions might begin with a focus on students independently performing computations or manipulating exponential expressions disconnected from real-world contexts, in M104, students should:

- regularly encounter real-world tasks involving constant percent change (growth or decay) that require them to employ models and predict values. Common settings for this type of growth include population growth, "viral" memes or internet videos, infectious diseases, finance, and radioactive decay (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially naming their own assumptions and variables and defending their choice of model as much as possible (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their solution methods (LP 2).

Spending time computing or manually completing tables with exponential expressions should not be a focus of this badge. In fact, the opposite is true. In M104, students should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models. For example, students should:
    - explain the real-world meaning of an exponential model.
    - compare two different exponential models.
    - explain the relationship between parameters in an exponential model and features of its graph.
    - use an exponential model to justify a claim about a real-world context.
    - reason about the differences between exponential and linear models (LP 3).

- have frequent opportunities to use reasoning to relate the algebraic form of an exponential function to its graph (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Instead of focusing only on the applications typically thought of as associated with exponential functions (e.g., bacterial growth), students should also be given opportunities to understand and reflect on the ways that exponential modeling can authentically involve real-world situations (LP 6). Some contexts to consider include:

- changes in revenue over time.
- the disproportionate impact of COVID-19 on people living in different zip codes.
- the human impact on the environment.

Often, coursework with exponential functions is focused on performing computations by hand. Instead, students should be given frequent opportunities to use spreadsheets and graphing software to allow for focus on understanding, rather than computation or symbolic manipulation. Students should write spreadsheet formulas to model phenomena that exhibit exponential growth or decay (LP 7).

Evidence of Learning

In M104 Modeling with Functions of Exponential Type, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
104.a: Engage in the modeling cycle Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).

Content and Practice Expectations	Indicators Choose an artifact where you...
104.b: Interpret exponential functions that arise in applications in terms of the context.	i. select two different data points that illustrate something interesting or important about the context you are modeling and describe what these points tell you about the topic.
	ii. use a model to compute output values and then interpret the values in the context of the problem.
	iii. describe the domain of the exponential function, including anything noticed about values that are not part of the domain or why the shape of the model might change for particular values of the domain.
	iv. describe the contextual meaning of the average rate of change between two points for an exponential function.
104.c: Analyze exponential functions using different representations.	i. create an exponential graph to show a particular context and describe important values that help to understand something about the problem you are exploring.
	ii. explain how the parameters of an exponential model relate to its graph.
104.d: Build an exponential function that models a relationship between two quantities.	i. identify the variables of interest in a given situation or data set that are interesting and that demonstrate an exponential relationship.
	ii. build a function that models the exponential relationship between two quantities in a situation.
104. e: Construct and compare linear and exponential models, solve problems, and draw conclusions.	i. distinguish between situations that can be modeled with linear functions and those that can be modeled with exponential functions.
	ii. provide insight into why some relationships are linear and others are exponential or explain why, in terms of a situation being modeled, an exponential function is a better model than a linear function.
104.f: Interpret expressions for exponential functions in terms of the situation they model.	i. describe the meaning of the parameters for an exponential function in terms of the context.
104.g: Summarize, represent, and interpret data on two quantitative variables for linear and exponential model fits.	i. choose a particular representation of your data to communicate something important to a particular person or audience and explain why you made that choice. In this badge, students are encouraged to investigate patterns of association

Content and Practice Expectations	Indicators Choose an artifact where you...
	in bivariate data, which includes the informal description of the fit of the curve and addresses the usefulness of the model for the particular context.
104.h: Understand the relevance of modeling with exponential functions.	i. recognize something that surprised you or that you learned about the topic through exponential modeling of data. Share what surprised you or what you learned.
	ii. highlight something you are most proud of in your learning about exponential modeling.
	iii. explain how modeling can be used to answer important questions.
104.i. Use functions of exponential type to determine values of interest in a real-world problem.	i. determine one or more output values for specific inputs to the function.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M104 Modeling with Functions of Exponential Type (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

The original Superman movie came out in 1978 and made about $300 million at the box office. If the yearly inflation rate is about 4%, how much money would this be equivalent to in 2022?

Sample Learning Experience

Students are given time to consider the question and develop an answer independently, before having an opportunity to share their response with others. Eventually, responses—some of which could be partially complete—that highlight multiple entry points and solution pathways are selected for sharing with the whole class. As you monitor students, look for approaches where students:

- generate a table and use differences to test for a linear fit.
- generate a table or expression where students use a two-step process to model exponential growth, such as $300 + 300(.04)$.
- generate a table or expression where students use the growth rate to model exponential growth.

One sample table is illustrated below:

Year	Revenue (millions of dollars)
1978	300
1979	312
1980	324.48

One sample expression is illustrated below:

$$300 \cdot 1.04^{t}$$

Instead of quickly asking students to identify an answer or approach, give students time to share responses to questions such as:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*
- *What additional questions could be answered about this context using some of the models we've seen? Are some models more helpful than others in answering these additional questions?*

Following the discussion, students are given an opportunity to reflect on and revise their work. Finally, one or more reasonable models are revealed explicitly, and students have an opportunity to reflect on the reasonable models and revise their work.

In this example, students are:

- given a cognitively demanding task. It requires that students model exponential growth as they determine the equivalent movie revenue in "today's dollars," but does not specify a particular method. To increase the cognitive complexity of the task, it is possible to incorporate more aspects of the modeling cycle. For example, there might be more emphasis on "interpreting the results" by asking students to consider a particular audience that might be interested in the results of such a problem. All learners are supported as the task supports peer learning: students have an opportunity to share and discuss each other's responses, followed by time to reflect and revise their own work (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- encouraged to have agency, as they choose and reflect on and refine their approaches. We can further increase students' agency in learning by offering a more open task (LP 4).
  For example, tasks can be less defined and require more assumptions on the part of the modeler: *A 30-year-old movie made about $300 million at the box office when it came out. How would you compare this amount of money to the revenue from a movie that came out today*?

Example 2

Students are given a table that shows the number of positive COVID-19 cases and the weekly rate of increase of infections for different US ZIP codes in April 2020:

ZIP Code	Number of Infections	Rate of Increase (4/2/20 - 4/9/20)
10025	343	5.6%

10029	462	26.4%
10035	56	3.2%

Sample Learning Experience

Students are asked to work in triads to use a Google Sheet (or other spreadsheet software) to model what the number of infections will be on July 1, 2020.

As students work, they respond to questions such as:

- *What formulas are you using in your Google Sheet? Write them as Google Sheet formulas and algebraic equations.*
- *What assumptions are you making as you develop these formulas?*
- *Use online search tools to locate census data to learn more about these ZIP codes. What do you notice? What do you wonder?*
- *Look up the actual rate of infection in each ZIP code on July 1, 2020. How accurate was your model? How would you change it? What additional assumptions or parameters might have caused the model to change?*
- *What does the work you have done suggest about the need to have people who are constantly studying data and refining models as part of their work?*
- *What additional questions surface for you as a result of the work above?*
- *How could the work you have done be used to inform policymakers, such as health or government officials?*

Following the discussion, students are given an opportunity to reflect on and revise their work. Finally, one or more reasonable models are revealed explicitly, and students have an opportunity to reflect on the reasonable models and revise their work.

In this example, students are:

- given a cognitively demanding task. It requires that students model exponential growth as they estimate the number of COVID-19 infections in different ZIP codes. Students are also asked to consider improvements to their model. They further engage in thinking about why models change as the parameters and assumptions change (LP 1).
- set up to have social interactions, as they work in triads to model the data (LP 2).
- able to explore areas of interests and develop a lens for aspirations, as they consider the work of an epidemiologist (LP 4).
- applying mathematics in a relevant, real-world situation. Students will notice how factors like population density and demographics correlate with infection rates, leading to considerations of policies that could mitigate the disproportionate impact of the pandemic on the population (LP 6).
- able to use technology as a tool, as they employ Google Sheets to perform their analysis (LP 7).

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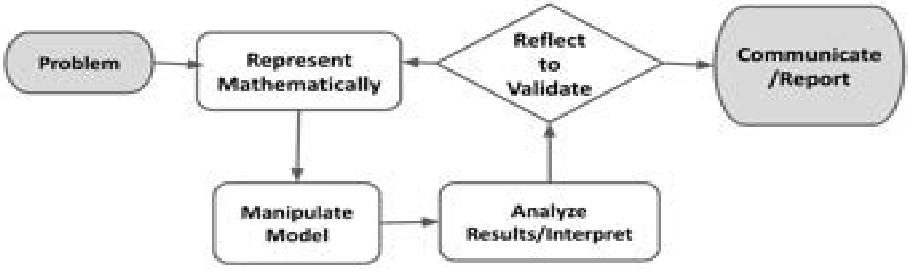
Safir, S., & Dugan, J. (2021). Building A Performance Assessment System. In Street Data (pp. 131-137). Thousand Oaks, CA: Corwin, A SAGE Company.

M111 Modeling with Data: One-Variable Measurement Data

Badge Catalog Description Some questions have a precise answer. For example, “How many years old is my little brother?” Other questions do not lead to a single answer, but to a set of data. In particular, some important questions about the world around us involve one-variable measurement data. This includes questions such as: “How much carbon dioxide is emitted daily in New York City?” or “Does life expectancy in one country differ meaningfully from that in another country?” By allowing us to analyze such questions, the tools and methods of statistics with one-variable measurement data can help us better understand our world, address the challenges of our time, and advocate for change. In M111 Modeling with One-Variable Measurement Data, you will pose and analyze meaningful statistical questions that yield one-variable measurement data. You will strategically use data displays and quantitative methods to draw conclusions, gain insight into the situation, and generate new questions. As you learn modeling with one-variable measurement data, you will use technology to create and analyze histograms and other visual displays. You will summarize data sets with measures of center, spread, and reason, interpreting the meaning of differences in center, shape, and spread. Finally, you will model data with normal distribution and estimate population percentages. The work for this badge builds a foundation for the future study of topics such as modeling with two-variable data, statistical inference, and data science. Modeling with one-variable measurement data is useful for careers in a variety of fields, like statistics, economics, biology, and computer science. Suggested prerequisites for this badge: understanding of fractions, decimals, and percentages; comfort with using formulas.

The M111 badge integrates mathematical modeling as an essential component of how students engage with one-variable measurement data. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M111 center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with one-variable measurement data, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M111 badge.

M111 Content and Practice Expectations

111.a	Engage in the modeling cycle.
111.b	Develop statistical questions in the course of modeling with one-variable measurement data.
111.c	Interpret differences in shape, center, and spread in the context of data sets, accounting for the possible effects of extreme data points (outliers).
111.d	Use measures of center and variability to describe one-variable measurement data.
111.e	Generate a random sample in the course of modeling with one-variable measurement data.
111.f	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
111.g	Estimate population percentages by using a normal curve to approximate a data distribution.

Learning Principles

In M111 Modeling with Data: One-Variable Measurement Data, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home

lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M111

In a typical instructional unit on one-variable statistics, students might engage heavily with lists of numbers either devoid of real-world meaning or associated with an artificial context. (For example, students might be asked to find the median of {1,1,2,3,3,3,3,4,4,5,5,5,5,5,5} without any description of what these numbers are meant to signify.) In contrast, in M111 students should understand measures of center and variability as tools that aid in statistical modeling; they should reason, calculate, and interpret using real data as much as possible.

Specifically, students in M111 should:

- regularly encounter real-world contexts and be given opportunities to develop and investigate statistical questions (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially using tools such as measures of center and variability, data displays, and sampling to answer questions about the world (LP 1).
- be able to choose questions and tasks that are organized around different scientific, social, or other topics that connect to one-variable measurement data, allowing students to have agency in their learning (LP 4).

- frequently collaborate and share their questions, models, calculations, and conclusions (LP 2).

Typically, in a course on one-variable statistics, students might be asked to repeatedly apply rules for computing the mean, median, or mode. In fact, the opposite should be true. In M111, students should develop understanding of the meanings of measures of center and variability and have significant experience interpreting them and using them as modeling tools. Students should spend as much time understanding and working with measures of center as they do with measures of variability. Similarly, instead of unduly focusing on the technical aspects of drawing data displays manually, students should spend ample time interpreting, analyzing, and reasoning about box plots, histograms, and other data visualizations.

Specifically, students in M111 should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning. For example, students should:
    - draw inferences and make comparisons based on measures of center and variability;
    - explain the contextual meaning of measures of center and variability;
    - describe characteristics of data sets by examining visual representations;
    - choose appropriate measures of center and variability and explain their choice;
    - use a visual data display to draw conclusions about populations (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Instead of focusing only on the applications typically associated with statistics in school (e.g., height or test score distributions), students should also be given opportunities to understand and reflect on the ways that statistical modeling can authentically involve real-world situations and create compelling arguments for change (LP 6). Some contexts to consider include:

- income inequality, including the racial wealth gap;
- climate change;
- racism and other forms of marginalization in social policy and other fields.

Often, coursework with one-variable statistics is focused on performing computations by hand. Instead, students should be given frequent opportunities to use spreadsheets and other software to aid in calculating measures of center and variability. In addition,students should use programs that can generate line plots and other data displays to allow students to focus on understanding, explaining, and reasoning about data (LP 7).

Evidence of Learning

In M111 Modeling with One-Variable Measurement Data, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
111.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
111.b: Develop statistical questions in the course of modeling with one-variable measurement data.	i. develop a statistical question to investigate.
	ii. explain or justify why your question is statistical.
111.c: Interpret differences in shape, center, and spread in the context of data sets, accounting for the possible effects of extreme data points (outliers).	i. determine and construct an appropriate graphical representation of collected data, such as box plot, line plot, histogram, or other data display.
	ii. describe the shape, center, and spread of a data set.
	iii. interpret the real-world meaning of the shape, center, and spread of a data set.
	iv. identify and explain the effect of an outlier on the shape, center, and spread of a data set.
111.d: Use measures of center and variability to describe one-variable measurement data.	i. find the mean or median of a set of data and explain its meaning.
	ii. find the interquartile range or standard deviation of a set of data and explain its meaning within the context of the data.
111.e: Generate a random sample in the course of modeling with one-variable measurement data.	i. describe a process for generating a random sample to investigate a statistical question.

	ii. explain why a proposed process generates a random sample that is appropriate for the statistical question being investigated.
111.f: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.	i. construct a box plot, line plot, histogram, or other data display for each of two related data sets.
	ii. find measures of center (mean or median) and variability (standard deviation or interquartile range,) for two data sets.
	iii. compare two data sets in terms of their center and variability.
	iv. explain the choice of measures of center and variability used to compare the two populations.
111.g: Estimate population percentages by using a normal curve to approximate a data distribution.	i. explain why a normal distribution is an appropriate approximation for a data set.
	ii. find an estimated population percentage by using a normal curve to approximate a data distribution.
	iii. explain the meaning of an estimated population percentage.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M111 Modeling with Data: One-Variable Measurement Data (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

Part 1: Making sense of student voice

Do you feel like you are given space to use your voice throughout the school day? How often? Do you wish you could talk more? Or less? Think about how you would answer the following questions:

- *How often do you get to talk in each of your classes? How would you quantify/track this?*
- *Who do you listen to? How much?*
- *How much do students talk throughout a typical school day?*

Part 2: Using our voice

For this task, you will have an opportunity to learn more about the student experience at your school. Your job is to prepare a presentation, in any form, that represents your answer to the question "What do you wish more teachers knew about the opportunities that students are given throughout the school day to use their voice?"

For this work, you will need to determine the following:

- *What data will help you tell the story? How would you quantify this?*
- *What statistical questions will you pose to get the data you need?*
- *What data displays are appropriate for your data?*
- *Who is the audience for this data?*
- *What form of a presentation (video, slideshow, report, poem, song, etc.) will work best for this purpose?*

Your presentation should include your response to "What do you wish more teachers knew from these data about the student experience?" supported by the evidence you collected.

Sample Learning Experience

In this experience, students will gather data to quantify the student experience within the context of how students use their voice across their classes. This creates space for students to recognize patterns and mathematically analyze these patterns to shed light on the question "*What do you wish more teachers knew about the opportunities that students are given throughout the school day to use their voice?"*

In preparation for Part 2 of this task, instructors must take into account their local context. They can select 2-3 courses that most students are currently enrolled in, such as English language arts or math and/or science, and plan for building consensus on the statistical questions they will use to collect data in those classes. As part of representing the data, students will make life-sized number lines. Based on the number of courses for which data will be generated, instructors should determine the number of life-sized number lines that will be needed so that each course has its own data display. The length of the class period should be used to determine the range and scale of numbers represented on each number line.

Part 1: Making sense of student voice
Use the questions posed to elicit student thinking about this topic and to design a collective approach to gathering data that helps the class see trends in how student voice is heard across the school day. Give students quiet time to reflect and respond to the questions posed. Then invite students to share with group mates to recognize similarities and differences in how they think about this topic.

As students share with team mates, monitor and select students to share their ideas. Record ideas that surface. The ideas generated through this share out will be used to design Part 2. Work to identify the question(s) and type of data students will gather in previously selected courses (English language arts, math and/or science). For example, the class may choose to count the number of times students are given a chance to talk during class in each of those classes, or students may work to count the total number of minutes students have an opportunity to talk during class in each of the classes.

As students work to agree on the data they want to collect, give time to reflect, using the following questions:

- *What might we learn from this data?*
- *What might be the limitations?*
- *How might we collect this data?*
- *How might we record this data? What would that look like?*

Before the end of the session, establish a date by which students will bring in their data.

Part 2: Using our voice
Before class begins, create the number lines along one side of the classroom using painter's tape on the floor or along the wall. During the learning experience, students can use sticky notes to show where they would place their data point along each number line to reflect the number of minutes they spend talking during each of the courses represented.

Once all students have represented their data on each of the number lines, do some informal analyses. Based on the distribution they see, ask students: *What story does this data tell?* Give students quiet time to think and formulate their responses. Then give students time to share with a partner. Monitor and select a couple of students to share their story with the class.

Invite students to choose a data display they would like to use to tell their story. Each student will construct a data display (digital or on paper) to represent the data collected during their activity. Once they are finished, have students find one other person with a different display type than their own and swap for analysis. As students discuss with their peers, monitor and select a few students to share key ideas that surfaced from their discussion. Listen for insights students gained about how students used features of their displays to tell the story they envisioned. Consider documenting noteworthy insights in a public space for future reference.

Have students study their peer's display and analyze using multiple interpretation questions. All answers vary depending on the data collected. For example, if students represented the story, "Most students talk about 7 minutes in a class," here are some sample questions to ask:

- Where in this display do we see that "most students talk about 7 minutes in a class"?
- What additional information can you glean from this display?
- What additional questions does this data surface for you?

Extension: Consider having students divide into two or more groups and choose their own question to explore and create a data set using a display choice they had not used before. For example, if they used a dot plot in the activity, they can then use a box plot for the choice question.

Give space for students to reflect on their work:

- *What story does the data tell?*
- *How does the display you choose work to tell the story?*
- *What was challenging about gathering this data?*
- *If you could do this again, what would you do differently? Why?*

Following the discussion, students are given an opportunity to reflect on and revise their work. Finally, one or more reasonable models are revealed explicitly, and students have an opportunity to reflect on the reasonable models and revise their work.

In this example, students are:

- set up to have social interactions, as they move around the learning space to create life-sized data displays and engage in conversations about what the displays communicate about the issue they are unpacking (LP 2).
- encouraged to have agency, as they engage in a task that asks them to consider the amount of agency they have during the school day. They are further asked to consider which audience would be most interested in the findings from their data collection to make choices about which data displays and presentation modes would be most effective in communicating the results to their target audience (LP 4).
- exploring an issue that likely has great relevance to them as it relates to understanding the extent to which the learning environment is constructed to value their insights and create space for them to engage effectively in their own learning via having their voices heard (LP 6).

Example 2

Students are given a problem like this:

Data can be used to gain insights about communities and help us understand issues.

Framing Questions
What are some questions you have about your community that could be answered by collecting data?

- *Why are these questions/issues important?*
- *Why should we care about this question/issue?*
- *Who else might care about these questions/issues?*
- *Why might it be important to them?*

For this task you will prepare a presentation in any form (slideshow, report, song, artwork) delivering a message you want to share on a topic of your choice, supported by the evidence you collect.

(Adapted from Julia Aguirre)

Sample Learning Experience
Invite students to write down questions they have about their community, such as:

- *How many cars does the typical family in my neighborhood own?*
- *How many children are there in a typical family in my neighborhood?*
- *How many doors does the typical house/apartment have in my neighborhood?*

As students write down ideas, circulate and look for trends. Give students time to connect with a partner and share what they generated. Select students to share ideas about their thinking with the class and revisit frequently the set of framing questions to help students connect data collection to understanding issues in communities. Offer students opportunities to go through iterative cycles of defining their topic and survey questions to use to gather data.

Students will collect data from their classmates and create a data display using technology (Desmos, applets, graphing calculator) and use summary statistics to describe what "typical" values are for the data they gathered.

Once students summarize their collected data, they will consider questions about generalizability of their data. These questions can be:

- Will all neighborhoods in your city have the same or similar data distributions? What do you expect to be the same? Different?
- Will all towns in your state have the same or similar data distributions? What do you expect to be the same? Different?
- Synthesis: What is the largest population that you could generalize to using your data? What about your data allows you to generalize to this new, larger population?

Students will then choose a nearby community, gather data from that community, and create a data display for data from the neighborhood and from a nearby community, using technology (Desmos, applets, graphing calculator). Once data sets are created for both communities, have students consider the following:

- How do the data distributions of both communities compare? [emphasis on writing answers in the context of the question of interest]
- Synthesis: Based on the differences you observed between the two communities, explain a method to gather data for your state that would be representative of the entire state.

Following the discussion, students are given an opportunity to reflect on and revise their work. Finally, one or more reasonable models are revealed explicitly, and students have an opportunity to reflect on the reasonable models and revise their work.

In this example, students are:

- given a cognitively demanding task. It requires that students engage in the modeling cycle as they determine statistical questions and gather and analyze data. Students are also asked to consider how the data they collect would compare to data from other cities and possibly the state, giving space to reflect on sampling methods (LP 1).
- set up to have social interactions, as they work with partners or small groups to model the data (LP 2).
- building conceptual understanding through reasoning about the application of findings from their data set to other populations. Students have opportunities to reason and justify their conclusions (LP3).
- able to explore areas of interests as they gather data about their communities and reflect on their findings (LP 4).
- engaging with mathematics as a human endeavor used to gain insight about their community and reflecting on the generalizability of their findings to other communities, thereby broadening ideas about what mathematics is, how it is used, and who it is for (LP 5).
- applying mathematics in a relevant, real-world situation. The context for engaging with mathematics is defined by the student, thereby authentically involving the students' real world (LP 6).
- able to use technology as a tool, as they employ Google Sheets, Desmos, or Geogebra to perform their analysis and design their data displays (LP 7).

Example 3

Students are given a problem like this:

Part 1:
Transportation engineers help determine the speed limit for any given road.

Assume you were hired as a transportation engineer.

- *What are some things you would want to consider in setting a speed limit?*

One of the criteria for setting a speed limit is the "85th Percentile Speed." Learn more about this idea by conducting additional research. One possible site to explore is this website: Understanding the 85th Percentile Speed by retired civil engineer Charlers Marohn. As you research, consider:

- *How would you explain what the 85th Percentile Speed criteria for setting speed limits is?*
- *Why is this issue important?*
- *Why should we care about this issue?*
- *Who else might care about this issue?*
- *Why might it be important to them?*

Part 2:
Your job is to inform the local community about the speeds on the road in front of your school.

- *Construct a plan to gather data, then find the mean and standard deviation of speeds on a road in front of your school.*
- *Use what you learned about the 85th Percentile Speed to examine what you notice about your community. What thoughts do you want to offer community stakeholders?*
- *Decide how to best share your message, supported by the evidence you collect.*

Sample Learning Experience

Launch this task by inviting students to write down questions they have about drivers in their community, such as:

- *How fast should cars be allowed to travel?*
- *What are some considerations in designing a reasonable speed limit for a particular road?*
- *How would you know if the speed limit you set is appropriate?*

Give space for students to share questions they generated with their classmates. Amplify questions that connect to the central focus of this prompt and introduce the Part 1 of the prompt above.

Provide quiet work time for students to consider their responses and offer opportunities for students to share their initial ideas with classmates before eliciting ideas to be shared with the whole class. Monitor and select students to share their preliminary thoughts with the whole class. Capture ideas that surface on a shared space, like chart paper or whiteboard, for students to be able to reference as they continue to explore.

Once students land on a shared understanding of the 85th Percentile Speed criteria for determining speed limits, introduce Part 2 of the prompt. Provide time for students to work in small groups to determine how they might go about collecting the data they need. Monitor as student teams puzzle, research, and design a plan. Invite student groups to share their plans. Invite engagement and reflection of each group's approach using questions such as:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think this approach is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect and revise their approach.

Once their plan for collecting data is finalized, give students time to gather and analyze the data they collect. Provide access to apps, such as Desmos, Geogebra, or spreadsheets, to help with the analysis.

Consider having students present their findings in a form that makes sense to them: a public service announcement, poster, song, report, using slideshow and/or audio/video recordings.

Give students an opportunity to reflect and revise their work as needed.

In this example, students are:

- set up to have social interactions, as they have opportunities to exchange ideas and partner with classmates (LP 2).
- encouraged to have agency, as they engage in a task that asks them to recognize and reflect on the driving habits of individuals in their community. They are further asked to consider which audience would be most interested in the findings from their data collection to make choices about which data displays and presentation modes would be most effective in communicating the results to their target audience (LP 4).
- developing an appreciation of mathematics as a human endeavor, one in which they feel a sense of belonging, and that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for (LP5).
- exploring an issue that likely has great relevance to them (LP 6).
- able to use technology as a tool, as they employ Google Sheets, Desmos, or Geogebra to perform their analysis and design their data displays (LP 7).

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M112 Modeling with Data: Two-Variable Measurement Data

Badge Catalog Description

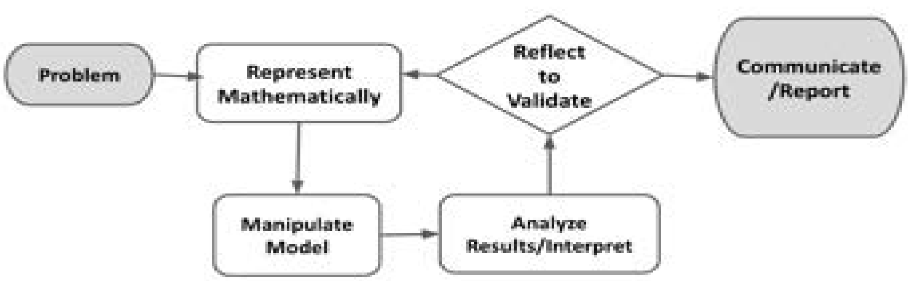
Some important questions about the world around us involve possible relationships between two variables. For example, we might want to know not only how life expectancy and poverty rates differ from one country to another, but also how national life expectancy correlates with national poverty levels. By allowing us to analyze such questions, the tools and methods of statistics with two-variable measurement data can help us better understand our world, address the challenges of our time, and advocate for change.

In M112 Modeling with Two-Variable Measurement Data, you will pose and analyze meaningful statistical questions that yield two-variable measurement data. You will reason with scatter plots, equation models, and quantitative methods to draw conclusions, gain insight into each situation, and generate new questions. As you learn modeling with two-variable measurement data, you will use technology strategically to create and analyze scatter plots to investigate patterns of association between two measured quantities, creating and reasoning with linear equation models where appropriate. The work for this badge builds a foundation for the future study of topics such as statistical inference and data science. Modeling with two-variable measurement data is useful for careers in a variety of fields, like statistics, psychology, biology, and medicine.

Suggested prerequisite for this badge: M101 Linear Equations: Concepts and Skills.

The M112 badge integrates mathematical modeling as an essential component of how students engage with two-variable measurement data. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M112 center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with two-variable measurement data, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M112 badge.

M112 Content and Practice Expectations

112.a	Engage in the modeling cycle.
112.b	Use scatter plots to represent two-variable measurement data.
112.c	Describe visible patterns between two data sets in a scatter plot.
112.d	Informally fit a straight line for scatter plots that suggest a linear association.
112.e	Fit a linear, quadratic, or exponential function to two-variable measurement data.
112.f	Use functions that have been fitted to two-variable measurement data to answer questions about the relationship being modeled.
112.g	Assess the fit of a function using the correlation coefficient, residuals, and other tools.

112.h	Explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data.

Learning Principles

In M112 Modeling with Two-Variable Measurement Data, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M112

In a typical instructional unit on two-variable statistics, emphasis may be placed on students' ability to fit a line to plotted points and memorize the phrases "positive correlation," "negative correlation," and "no correlation." In many cases, students apply these skills to scatter plots that may not have any indicated contextual meaning. By contrast, in M112 students should come to see fitting lines and describing patterns of association as valuable tools for modeling; they should reason, calculate, and interpret using real data as much as possible. Specifically, students in M112 should:

- regularly encounter real-world contexts and be given opportunities to develop and investigate statistical questions (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially using tools such as scatter plots and curve-fitting to answer questions about the world (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics that connect to two-variable measurement data, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their questions, models, calculations, and conclusions (LP 2).

In a typical course on two-variable statistics, students might focus on the technical aspects of constructing scatter plots manually or calculating correlation coefficients by hand. Instead, in M112, students should spend ample time interpreting, analyzing, and reasoning about scatter plots, using curves fitted to the data when appropriate. Specifically, students in M112 should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning. For example, students should:
    - describe patterns visible in scatter plots;
    - explain the meaning of patterns of association, with care to distinguish between correlation and causation;
    - informally fit straight lines to data;
    - use curves that have been fitted to data to answer questions about the relationship being modeled;
    - reason about the appropriateness of models using correlation coefficients or residuals;
    - explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Instead of focusing only on the applications typically associated with statistics in school (e.g., correlations between height and armspan or level of education and income), students should also be given opportunities to understand and reflect on the ways that statistical modeling can authentically involve real-world situations and create compelling arguments for change (LP 6). Some contexts to consider include:

- the relationship between industrial practices and climate change;
- factors associated with reduced access to high quality health care.

Students should be given frequent opportunities to use software to aid in calculating correlation coefficients, constructing scatter plots, and curve-fitting; these uses allow students to focus on understanding, explaining, and reasoning about data (LP 7).

Evidence of Learning

In M112 Modeling with Two-Variable Measurement Data, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
112.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
112.b: Use scatter plots to represent two-variable measurement data.	i. construct (by hand or using technology) a scatter plot from collected data.

Content and Practice Expectations	**Indicators** **Choose an artifact where you...**
	ii. explain the meaning of a data point plotted on a scatter plot.
112.c: Describe visible patterns between two data sets in a scatter plot.	i. identify the type of association visible in a scatter plot.
	ii. give a hypothesis that might explain the association identified in a scatter plot.
112.d: Informally fit a straight line for scatter plots that suggest a linear association.	i. sketch a line that follows a pattern of linear association.
	ii. justify location and direction of a line fitted to data.
112.e: Fit a linear, quadratic, or exponential function to two-variable measurement data.	i. graph a linear, quadratic, or exponential function that models two-variable measurement data.
	ii. find the equation for a linear, quadratic, or exponential function that models two-variable measurement data.
	iii. justify the choice of linear, quadratic, or exponential function to model two-variable measurement data.
112.f: Use functions that have been fitted to two-variable measurement data to answer questions about the relationship being modeled.	i. explain the meaning of parameters for a function fitted to two-variable measurement data.
	ii. use a function fitted to two-variable measurement data to predict values.
	iii. describe any limitations to the model relative to answering particular questions of interest.
112.g: Assess the fit of a function using the correlation coefficient, residuals, and other tools.	i. find the correlation coefficient for a function fitted to two-variable measurement data.
	ii. explain the meaning of a correlation coefficient in the context of the problem.
	iii. plot residuals for a function fitted to two-variable measurement data.
	iv. interpret a residual plot in terms of a function's goodness of fit.
112.h: Explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data.	i. develop a proposal or recommendation grounded in your analysis.
	ii. explain a limitation of a curve fitted to data.
	iii. distinguish between correlation and causation.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples Modeling with Data: Two-Variable Measurement Data (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

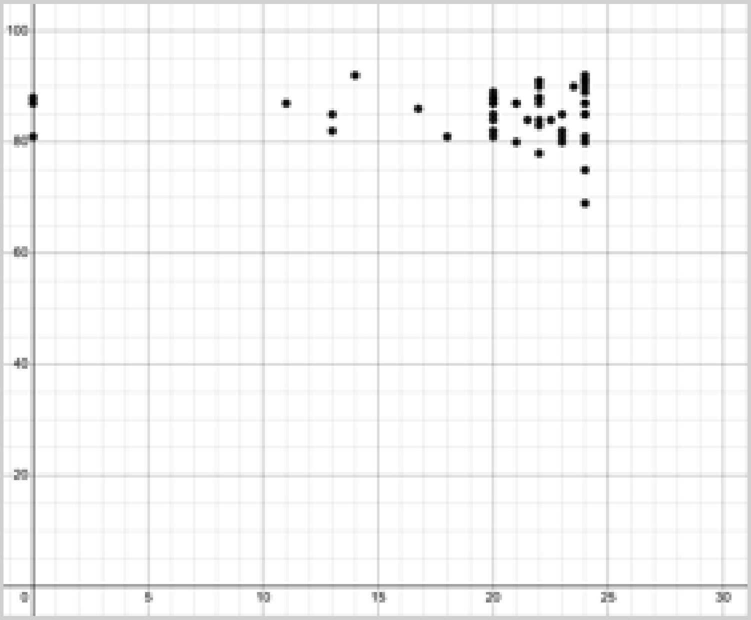
Example 1

Sample Learning Experience
Part 1: Making Sense of a Scatter Plot

Students are invited to consider graduation rates and the factors that might impact them, using discussion questions such as these:

- *What do you think a state's "graduation rate" is?*
- *What factors might cause a state's graduation rate to be higher or lower?*

After discussing, students are provided this scatterplot and told that each point represents a state's data on two variables. The response variable is the state's graduation rate.



To ensure students have a secure understanding of the scatter plot and the meaning of the response variable, share with students that Mississippi has a graduation rate of 85%. Ask students to examine the graph and identify a point that could represent Mississippi.

Students are then asked to brainstorm, first individually, then with a peer group, the answers to these questions:

- *What could be the explanatory variable?*
- *What type of association is represented by this data (linear, quadratic, power, etc.)?*
- *There are three data points on the y axis. What could these points represent?*

After sharing, students are told that the explanatory variable for the data above is the number of courses required by U.S. states in order to graduate from high school.

Students are then asked to brainstorm, first individually, then with a peer group, the answers to these questions:

- *Do you think the number of required courses is a good predictor for high school graduation rate?*
- *Now that you know that the response variable is high school graduation rate, what could the three data points on the y-axis represent?*

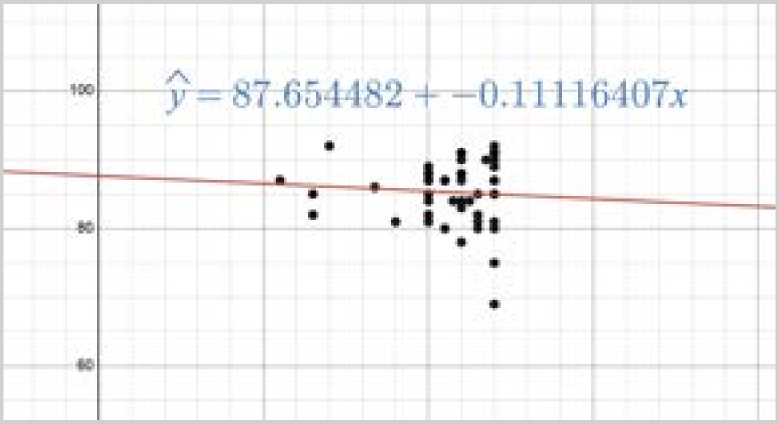
To deepen understanding of the context being modeled, students investigate the information found in the Education Commission of the States' comparison of each state's high school graduation requirements and the National Center for Education Statistics' comparison of state graduation rates. They respond to questions such as these:

- *What are the coordinates for a state like Alabama? What do the coordinates represent?*
- *What are the coordinates of Pennsylvania? What do the coordinates represent?*
- *How does the meaning of the coordinates of these points compare? Are they tracking the same things?*
- *When creating a mathematical model, it is important that the quantities being used are tracking the same things. Why would it make sense to exclude data from Pennsylvania, Colorado, and Massachusetts?*

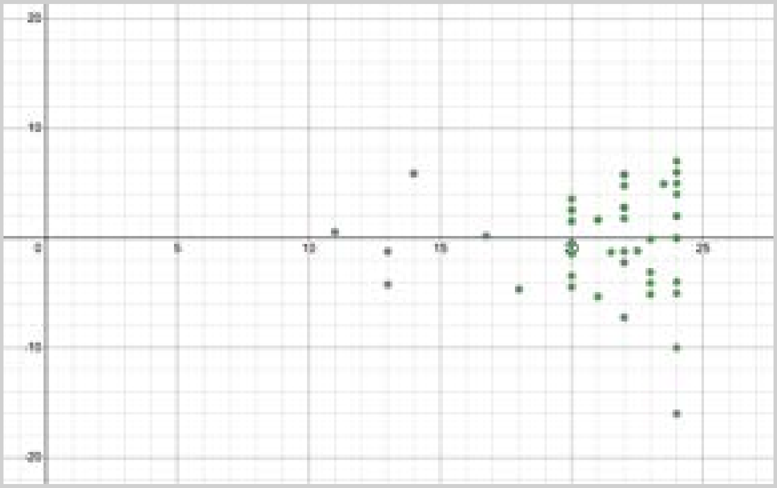
Part 2: Using a Line of Best Fit

Using technology, students are asked to estimate and draw in a line of best fit for the data set. (See, for example, this Desmos calculator link.) Students discuss and respond to these questions:

- *What does the y-intercept and slope of your line of best fit mean for the given context?*
- *Use technology to fit a regression model to the data. Find the value of the correlation coefficient, r. What does this value mean for the given context? (See for example, this Desmos calculator link.)*
- *Is a linear model a good fit for this set of data?*
- *Using technology, find the value of the coefficient of determination, r^2. The coefficient of determination shows the amount of variability accounted for in the model. Does the value of r^2 you calculated validate or dispute the claim that a higher number of required courses in high school leads to higher graduation rates?*
- *Here is the residual plot of data given above based on the linear regression model shown below:*



Residual plot:



Is a linear model appropriate for the given data?

Students are then asked to consider the other potential factors influencing graduation rates brainstormed initially in Part 1. Students find state level data and repeat the process to examine the effect of their chosen explanatory variable is a better predictor of graduation rates.

In closing, students are asked to consider and respond to these questions:

- *Which explanatory variable that you have examined impacts high school graduation rates more? Support your answer with appropriate statistics.*
- *What new insights have you gained about the number of course requirements and graduation rates?*
- *What recommendations would you offer or new questions surface for you?*

In this example, students are:

- given a cognitively demanding task. It requires that students make sense of a situation and a statistical model, in addition to generating their own model (LP 1).
- set up to have social interactions, as they are invited to brainstorm, discuss, and share ideas with one another. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).

- building conceptual understanding through reasoning, as they make and justify conjectures about the explanatory variable in a scatter plot (LP 3).
- engaging with a task that is relevant to their lives, focusing on graduation rates and requirements (LP 6).
- encouraged to use technology as a tool for understanding two-variable statistics (LP 7).

Example 2

Students are given a problem similar to this:

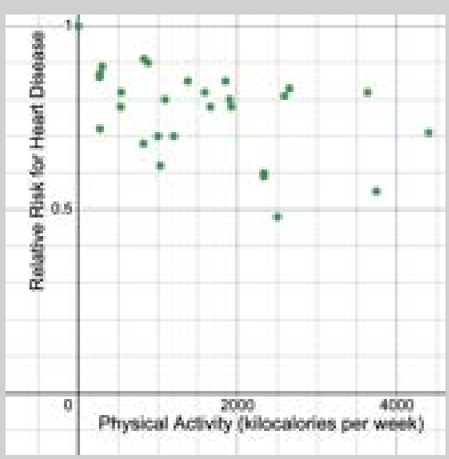
Part 1: Develop a Statistical Question

According to Harvard Health Publishing web article, "Heart Health", a healthy heart "beats about 2.5 billion times over the average lifetime, pushing millions of gallons of blood to every part of the body. This steady flow carries with it oxygen, fuel, hormones, other compounds, and a host of essential cells. It also whisks away the waste products of metabolism." When the heart can't perform at optimal health, it is said that a person has developed some form of cardiovascular disease or heart disease. Doctors have developed a way to quantify a person's risk for developing heart disease; it is called relative risk ratio.

1. *What are some factors that might lead to heart disease?*
2. *A relative risk ratio is a number that compares the possibility of getting heart disease among one group with the possibility of getting that disease in another group. If the risk ratio is 1, that means both groups have the same possibility of getting heart disease. Based on your current understanding, answer the following questions using a factor from your answer to Question 1:*
    a. *What do you believe are the two groups involved?*
    b. *What do you anticipate the relative risk ratio will be: about 1, greater than 1, or less than 1? How did you arrive at this conclusion?*
    c. *What are some statistical questions you can ask about heart disease?*

Part 2: Understand a Mathematical Representation

In one study, researchers set out to quantify the specific amounts of physical activity required for lower risks of heart disease. In this scatter plot, the explanatory (independent) variable is the physical activity measured in kilocalories burned per week by participants in various studies. The response (dependent) variable in the data set below is the relative risk of heart disease, when comparing physical activity during leisure-time with a group who has no leisure-time physical activity.



Based on: Dose response between physical activity and risk of coronary heart disease: a meta-analysis

3. *What do you notice? What do you wonder?*
4. *How might you use this graph to answer the statistical question, "Do higher levels of physical activity lower the risk of heart disease?"*

Part 3: Manipulate the Model

5. *Work with this representation of the data on Desmos Calculator: Physical Activity and Relative Risk for Heart Disease to determine the line of best fit.*
6. *Find the value of the coefficient of determination, r^2. Does the value of r^2 you calculated validate or dispute the claim that a higher level of physical activity lowers the risk of heart disease?*

Part 4: Interpreting the Model

7. *Do you think that a level of physical activity, quantified by kilocalories per week, is a good predictor of a relative risk for heart disease?*

Part 5: Reflection

8. *What other variables do you think contribute to heart disease?*
9. *Consider your list of variables. Which ones does your group believe could be a better predictor of a shortened lifespan due to heart disease?*

Sample Learning Experience

Part 1: Develop a Statistical Question

Students are primed with information about heart disease, including what it is and possible contributing factors. Students are asked to brainstorm factors that might decrease a person's risk for heart disease. Give students an opportunity to learn more about terms that surface. One term students may be wanting to explore is relative risk ratio; they can learn more about this at the website Principles of Epidemiology | Lesson 3 - Section 5.

Some examples of factors that might contribute to a person's risk for heart disease might include:

- personal habits such as smoking, exercise, and eating
- eating habits
- physical activity or exercise
- genetic factors
- environmental factors

Students then collaboratively develop a statistical question to explore heart disease. Some may wonder about the relationship between heart disease and walking speed, which may lead to a question like: Is a higher average walking speed associated with a lower rate of heart disease?

Consider having students post their statistical questions in a visible space for all to reflect on. Invite students to give each other feedback on the questions they developed in order to spark possibilities for moving forward.

Part 2: Understand a Mathematical Representation

For this part, students will work on making sense of the given data set and model. Time permitting, you may offer students time to do research and examine another factor related to the statistical questions they developed. Highlight statistical questions that lend themselves to the data given.

Students are asked to consider how they could use a scatter plot to explore the statistical question: “Do higher levels of physical activity lower the risk of heart disease?” First, give students time to think individually for two-three minutes. Next, have them turn and talk or speak within small groups.

Consider posing the following questions as students discuss their insights:

- *What is the explanatory (independent) variable? Is it a reasonable variable to consider for this context?*
- *What does the response (dependent) variable mean?*
- *What type of association is represented by this data (linear, quadratic, power, etc.)?*
- *There are multiple data points in the set. Choose one and explain what it means.*

Part 3: Manipulate the Model

Students use technology to draw a line of best fit for the scatter plot. Have students navigate independently to the Desmos Calculator: Physical Activity and Relative Risk for Heart Disease. Students respond to and discuss questions such as these:

- *How did you decide your line of best fit?*
- *What do the y-intercept and slope mean in the context of this problem?*
- *What type of model is best for this data set?*
- *How do you feel about using your model to explain the relationship between physical activity and risk of heart disease? What new questions do you have?*

Part 4: Interpreting the Model

Students consider and discuss this question, using their modeling, data, and analysis to support their responses. Encourage students to ask clarifying questions of each other, so that arguments are understood. Some questions to consider are as follows:

- *What about this reasoning makes sense to you?*
- *Is there anything you disagree with?*
- *What questions surface for you?*

Part 5: Reflection

Students consider further variables to explore by discussing the questions posed. Offer space for various ideas to surface. Provide time for students to reflect and revise their work throughout.

In this example, students are:

- given a cognitively demanding task. It requires that students go through the full modeling cycle to investigate the relationship between weekly physical activity and relative risk rate of heart disease (LP 1).
- set up to have social interactions, as they collaborate to develop a statistical question (LP 2).
- building conceptual understanding as they reason and reflect throughout this iterative process (LP 3).

- given opportunities to explore questions that surface as they make their way through this process, building a sense of agency (LP 4).
- applying mathematics in a relevant, real-world situation. Students are asked to consider a matter of public health. As they do, they have an opportunity to learn how researchers work to develop mathematical models to explain possible relationships between factors present in a phenomena (LP 6).
- encouraged to use technology to create a scatter plot and generate a line of best fit (LP 7).

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M113 Modeling with Probability

Badge Catalog Description

What makes a game fair? How can we predict the likelihood of a candidate winning an election or a team winning a championship? Modeling with probability is a powerful way to gain insight into the world around us and answer questions like these. Nature, society, business, and everyday life are full of situations involving uncertainty and randomness. Probability models enable us to go beyond guesswork and handle these situations quantitatively.

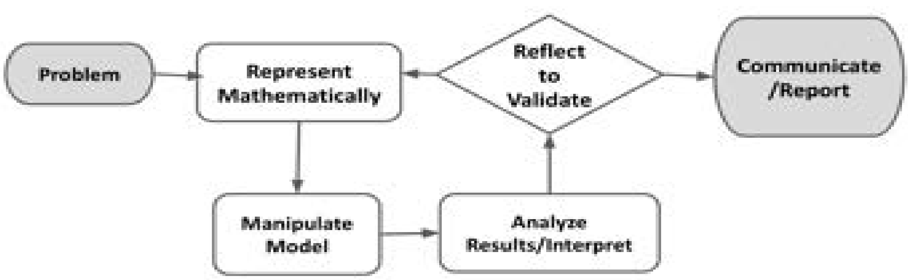
In M113 Modeling with Probability, you will calculate and estimate probabilities by using data. You will make sense of and interpret these results to make predictions, analyze strategies, and inform decisions. As you learn modeling with probability, you will strategically utilize tools like tables, tree diagrams, counting techniques, and the rules of probability. Using technology, you will simulate random processes, approximate probabilities, interpret results, and make appropriate decisions about everyday events. Constructing and interpreting two-way frequency tables to determine if events are independent, approximating conditional probabilities, and calculating the expected value to make informed decisions are also important topics of this badge. Modeling with probability is useful for careers in a variety of fields, like meteorology, risk management, nursing research, public policy, sports, and finance.

Suggested prerequisites for this badge: concepts of fractions, ratio, and percentages.

This badge is suggested as a prerequisite for: M211 Data Management and Visualization.

The M113 badge integrates mathematical modeling as an essential component of how students engage with probability. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M113 center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with probability, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M113 badge.

M113 Content and Practice Expectations

113.a	Engage in the modeling cycle.
113.b	Understand and evaluate random processes underlying statistical experiments.
113.c	Use a model to determine the probability of an event.
113.d	Use the rules of probability to compute probabilities of compound events.
113.e	Understand independence and conditional probability and use them for modeling.
113.f	Use probability to make decisions.

Learning Principles

In M113 Modeling with Probability, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M113

In a typical instructional unit, students often focus their time on using probability in fabricated, artificial settings; for example, students may work to understand and analyze situations involving spinners, dice, and cards. By contrast, in this badge, students should be given opportunities to model and reason with probability in ways that are relevant to understanding and analyzing events and situations in the world. Students should come to see the ways that probability can help answer questions of interest, such as "What is the likelihood that a particular candidate will win an election?" or "How likely is it that there will be a recession next year?" In M113, students should:

- regularly encounter real-world tasks that require them to develop probability models and predict values (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially naming their own assumptions and variables and defending their choice of model as much as possible (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their solution methods (LP 2).

In many typical courses, students are asked to treat probabilities procedurally, without full understanding of the real-world meaning of probabilities. Probabilities thus become mere symbols to manipulate. Spending time computing probabilities, with students being asked to simplify, compare, multiply, and add fractions, or to manually convert fractional probabilities to decimal representations, should not be a focus of this course. Instead, in M113, students should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models. For example, students should:
    - understand probabilities as the long-run relative frequency of an event;
    - relate probability models to simulations and chance experiments;
    - explain the real-world meaning of probabilities;
    - compare probabilities;
    - use a probabilistic model to justify a claim about a real-world context;
    - use probability to make decisions (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Instead of focusing only on the applications typically thought of as associated with probability (e.g., fabricated situations involving spinners, coins, or dice), students should also be given opportunities to understand and reflect on the ways that probabilistic modeling can authentically involve real-world situations (LP 6). Some contexts to consider include:

- meteorology;
- insurance and risk management;
- entrepreneurship and finance.

In this badge, students should be given frequent opportunities to use technology to simulate chance experiments to increase understanding (LP 7).

Evidence of Learning

In M113 Modeling with Probability, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
113.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
113.b: Understand and evaluate random processes underlying statistical experiments.	i. describe or use an experiment or simulation to generate frequencies for compound events.
	ii. decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.
	iii. make an inference about a population based on a random sample.
113.c: Use a model to determine the probability of an event.	i. determine the probability of an event by reasoning about the number of possible outcomes.
	ii. compare two different events in terms of their probabilities.

Content and Practice Expectations	Indicators Choose an artifact where you...
113.d: Use the rules of probability to compute probabilities of compound events.	i. determine the probability of the intersection or union of multiple events.
113.e: Understand independence and conditional probability and use them for modeling.	i. describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not")
	ii. explain or show why two events can be considered independent or dependent.
	iii. determine a conditional probability.
113.f: Use probability to make decisions.	i. use probability to make a recommendation.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M113 Modeling with Probability (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a task like this:

Part 1: The Titanic

On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Others believe that the survival rates can be explained by the "women and children first" policy. Data on survival of passengers are summarized in the table below. Investigate what might and might not be concluded from the given data.

	Survived	Did not survive	Total
Children in first class	4	1	5
Women in first class	139	4	143
Men in first class	58	118	176
Children in second class	22	0	22
Women in second class	83	12	95
Men in second class	13	154	167
Children in third class	30	50	80
Women in third class	91	88	179
Men in third class	60	390	450
Total passengers	500	817	1317

The Statistics of the Disaster

The Titanic 3, (Illustrative Mathematics, 2016)

Part 2: Extending the learning

1. *What additional questions surface for you as a result of the work above?*
2. *What are your thoughts and feelings regarding what you have learned about travel in 1912?*
3. *Are there any current events you believe to have similar disparities as those seen in the survival rates of the Titanic? Work to formulate a plan for gathering data about this event and conduct an investigation to see what might or might not be concluded from that data.*
4. *Based on your analysis, what conclusions have you reached? What would you like to see happen?*

Sample Learning Experience

Part 1: **The Titanic**
Launch this task by displaying the following information about the Titanic and passenger survival data. Give students time to study, and pose the questions: What do you notice? What do you wonder?

On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Data on survival of passengers are summarized in the table below.

	Survived	Did not survive	Total
First class passengers	201	123	324
Second class passengers	118	166	284
Third class passengers	181	528	709
Total passengers	500	817	1317

After some quiet time, give students an opportunity to share their ideas with a partner. As students share, listen for observations and questions that get at how class and survival rate might be related. Capture ideas that surface on a visible surface, like chart paper or white board. Leverage this part of the conversation to then transition to the prompt presented. Refer to student generated observations and questions as applicable.

As students work, notice various ways students make sense of the information and formulate a plan for responding to the prompt. Select some students or partner pairs to share their thinking with the class. Consider using questions below to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect on and revise their work. Use questions students generate to further engage students with this data, as well as to amplify the mathematics in play.

Part 2: **Extending the learning**
Give students time to work individually or with a small group to identify current events they'd like to investigate in this manner. Monitor and select student voices to share their preliminary progress, focusing the discussions on the formulation of their plans. Consider using the questions below to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect on and revise their work.

Give additional time for students to conduct their investigation and present their findings to the class or other audience they feel is appropriate.

In this example, students are:

- given a cognitively demanding task. It requires that students formulate a plan to answer the question at hand. Students also have the opportunity to apply the knowledge and skills from investigating the survival rates of the Titanic to a current event of their choice (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- deepening their understanding of conditional probability and independence as they work to make sense of data in two-way tables as well as when they work to gather data for Part 2 (LP 3).
- able to explore skills and dispositions around what it means to model with probability. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to make sense of the world (LP 4).
- developing a sense of belonging as they formulate and carry out an approach to the challenges presented. This offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- given insight into how statisticians work to interpret data and determine associated probabilities of events (LP 6).
- using technology as a tool, as they employ spreadsheet and/or graphing software to perform their analysis (LP 7).

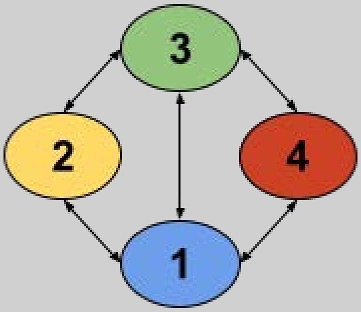
Example 2

Students are given a task like this:

In this task you will explore the connection between random walks and internet searches.

Part 1: Random Walks

Let's begin by exploring how random walks work. Consider the following map with four points of interest.

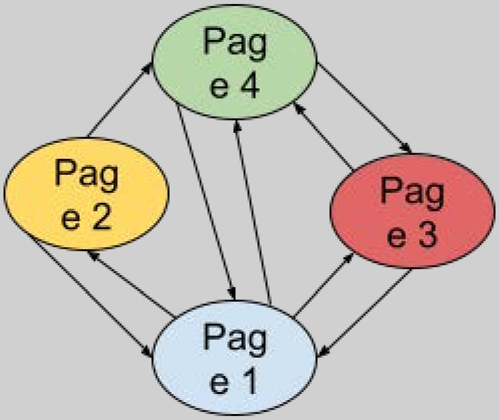


1. *If someone starts at one of the points and randomly decides where to go next, what do you anticipate will happen? Make at least three predictions.*
2. *Design and carry out a simulation to test one of your predictions. Determine ways to quantify how often someone can end up at one of the points.*
3. *What did you learn from designing a simulation to test your predictions?*
4. *What do you anticipate the random walk phenomenon has to do with internet searches?*

Part 2: Internet Searches

Watch this short video [The Internet: How Search Works] (Code.org., 2017)

1. *What connections do you see between the work on Part 1: Random Walks and internet searches?*
2. *Let's examine how internet pages are ranked by building on what you've learned about random walks. To simplify things, let's consider an internet with four pages that have links to each other as illustrated in the diagram below.*
    a. *What do you notice about the relationships between the pages? How do you anticipate the pages will rank?*



3. *Design and carry out a simulation to test your prediction. Based on the simulation, how do the pages rank? Does this surprise you?*
    a. *What is the relative frequency of visits to each page? What do these percentages mean?*
4. *In the video, we hear Akshaya explain how "spammers try to game the search algorithm." How might you modify the map above to illustrate how this works? Design a simulation as before and prepare to share your results.*
5. *Based on your current understanding, what might explain why people have different experiences when they search the internet? What are your thoughts on this?*

(adapted from Dr. Claudio Gómez-Gonzáles, Carleton College)

Sample Learning Experience

Part 1: Random Walks
Introduce the concept of random walks by displaying the map in Part 1: Random Walks. Ask students: what do you notice? What do you wonder? Record student responses on a whiteboard or chart paper to refer to as appropriate.

Ask students to consider what would happen if someone were to start at oval 1 and use something like a die or coin to decide which way to go next. Share with students that this process of randomly selecting which way to move is called a *random walk* in mathematics and has several applications to things like the physical phenomena called Brownian motion and searching the internet.

In this task, students will explore the connection between random walks and the process of searching the internet. Consider providing a 12-sided die and/or use of technology for generating random numbers.

Next, give students time to engage in prompts individually and in small groups. As students work, monitor for various ways students formulate a plan to respond to the prompt. Make note of how students attend to the various aspects of designing a simulation: number of trials, method for recording, and assignment of outcomes of a die, or how to use technology to determine the movement from node to node. Select some students or partner pairs to share their preliminary plans with the class. Consider using the questions below to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect on and revise their work. Give additional time for students to conduct their investigation and present their findings to the class.

Finally, share one or more reasonable models and give students an opportunity to reflect on the reasonable models and revise their work. Use student responses to Question 4 to launch Part 2: Internet Searches.

Part 2: Internet Searches
Share the video and give students time to discuss Questions 1 and 2 with group mates. Record connections and questions students share and use them throughout the discussion as appropriate. Amplify ideas that get at the connection between random walks and internet searches, such as the idea that we can devise a simulation to understand how ranking works. This is the focus of Part 2.

Next, give students time to engage in Question 3 individually and in small groups. As students work, monitor for various ways students formulate a plan to respond to the prompt. Make note of how students attend to the various aspects of designing a simulation: number of trials, method for recording, and assignment of outcomes of a die, or how use of technology, to determine the movement from node to node. Select some students or partner pairs to share their preliminary plans with the class. Consider using the questions below to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect on and revise their work. Give additional time for students to conduct their investigation and present their findings to the class. Repeat the process above for Question 4.

Finally, share one or more reasonable models and give students an opportunity to reflect on the reasonable models and revise their work. Consider having students design a public service announcement on the ideas they have shared in response to Question 5.

In this example, students are:

- given a cognitively demanding task. It requires that students formulate a plan to answer the question at hand. Students also have the opportunity to apply the knowledge and skills from making sense of random walks to make sense of how search engines work (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- deepening their understanding of probability and independence as they design and carry out a simulation. Students also work to interpret the results of their simulations in terms of the context (LP 3).
- able to explore skills and dispositions around what it means to model with probability. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to make sense of the world (LP 4).
- developing a sense of belonging as they formulate and carry out an approach to the challenges presented. This offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- given insight into how mathematicians use probability to model phenomena like internet searching (LP 6).
- using technology as a tool, as they employ spreadsheet and/or graphing software to perform their simulation and analysis (LP 7).

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M151 Modeling with Geometry

Badge Catalog Description

How much aluminum is needed to build a prototype robot? Which design for a new cell phone case will have the lowest cost? Modeling with geometry is a powerful way to gain insight into the world around us and solve design challenges. Whether we are laying out a backyard flower bed, advocating for a new neighborhood park, or drafting a blueprint for a building, geometric models hold great power to help us ask and answer questions about the physical world.

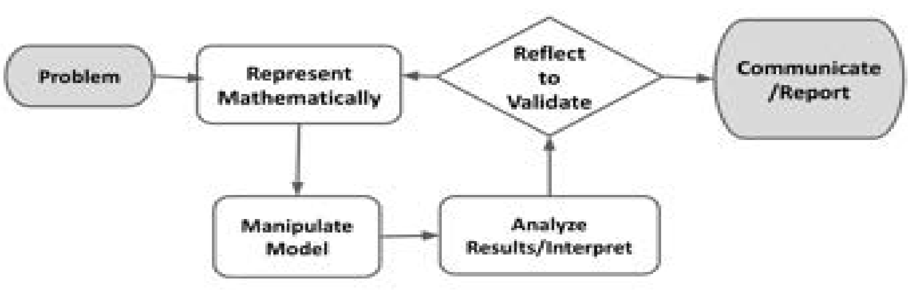
In M151 Modeling with Geometry, you will apply concepts of measurement to make sense of real-world situations while creating models that consider important situational features, such as cost and size constraints, or aesthetic considerations. As you learn about modeling with geometry, you will use technology strategically to model complex objects and scenarios. You will apply principles of length, area, volume, and angle to model geometric measurements and spatial relationships, and apply the Pythagorean Theorem to solve problems in real-world contexts. Modeling with geometry is useful for careers in a variety of fields, like the arts, engineering, and architecture.

Suggested prerequisites for this badge: comfort with middle-school-level problems involving length, area, volume, and angle measure; comfort with—or interest in learning about—writing and solving simple equations to solve problems; comfort with using formulas.

This badge is suggested as a prerequisite for: advanced high school courses in mathematical modeling.

The M151 Modeling with Geometry badge integrates mathematical modeling as an essential component of how students engage with geometry. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

The learning expectations for M151 Modeling with Geometry center on the CCSSM modeling cycle described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, Modeling, K-12, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with geometry, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M151 badge.

M 151 Content and Practice Expectations

151.a	Engage in the modeling cycle.
151.b	Use geometric shapes, their measures, and their properties to describe objects.
151.c	Apply concepts of density based on area and volume in modeling situations.
151.d	Apply geometric methods to solve design problems.
151.e	Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.
151.f	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

151.g	Solve real-world problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
151.h	Use the Pythagorean Theorem to solve right triangles in applied problems.

Learning Principles

In M151 Modeling with Geometry, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M151

The study of geometric figures like cylinders, pyramids, and circles is often taught with an emphasis on memorization of the formulas for volume or area and performing computations with these formulas. In M151, students should instead:

- collaboratively explore geometric figures, with opportunities to discuss and gain deep understanding of formulas for volume and area (LP 2).
- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning about geometric models (LP 2).
- explain formulas, developing logical justifications using different representations and modes of expression (LP 3).
- employ technology as a tool for viewing and understanding geometric figures, and also as a tool for computing (LP 7).

Furthermore, the study of geometric figures often involves a significant amount of work without context, as students apply formulas and perform computations without connection to the real world. Often, when problems are given in context, they center around a single computation and do not require the modeling cycle (see above). Instead, in M151, students should:

- engage with multiple parts of the modeling cycle (see above), especially naming their own assumptions and variables and defending their choice of model as much as possible (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- be able to make decisions and assumptions in developing their own models (LP 4).
- engage with models beyond mere computation, including
    - explaining assumptions that underlie a geometric model;
    - comparing two different geometric models;
    - critiquing and improving a geometric model;

    - using a geometric model to justify a claim about a real-world context (LP3).
- frequently collaborate and share models with each other (LP2).

Geometric models also present opportunities to understand and reflect on the ways that geometric modeling can authentically involve real-world situations (LP 6). Some contexts to consider include:

- the mathematics of congressional districts.
- product redesign.

Evidence of Learning

In M151 Modeling with Geometry, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations.

Content and Practice Expectations	Indicators Choose an artifact where you...
151.a: Engage in the modeling cycle Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
151.b: Use geometric shapes, their measures, and their properties to describe objects.	i. represent a context, situation, or object from outside mathematics as a mathematical object.
	ii. explain how the mathematical object(s) represent the real-life context, situation, or object.
	iii. use properties of the mathematical object to determine quantities of interest and interpret those values within the context.
151.c: Apply concepts of density based on area and volume in modeling	i. apply concepts of density based on area in modeling situations.

Content and Practice Expectations	Indicators Choose an artifact where you...
situations.	ii. apply concepts of density based on volume in modeling situations.
151.d: Apply geometric methods to solve design problems.	i. design an object or structure to satisfy physical constraints or features in a real-life context.
151.e: Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.	i. use informal justifications to explain or reason about circumference, area and volume formulas. Justifications may reflect dissection arguments, Cavalieri's principle, and/or informal limit arguments.
151.f: Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.	i. solve real-life and mathematical problems using volume formulas for cylinders, pyramids, cones, and spheres.
151.g: Solve real-world problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.	i. analyze the context of the problem to determine which measurable geometric attributes are important to reasoning about the problem.
	ii. solve real-life problems using area, volume, and surface area of two- and/or three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and/or right prisms.
151.h: Use the Pythagorean Theorem to solve right triangles in applied problems.	i. recognize situations where the Pythagorean Theorem is a useful tool for representing and solving problems.
	ii. solve applied problems using the Pythagorean Theorem.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M151 Modeling with Geometry (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a scenario like this:

Part 1: Design a Controller

You are a small business owner and you will create a new controller design for your favorite gaming system. You plan to sell your controller in an online store to reach a greater market.

1. *Sketch a design of the controller. Consider the shapes you'll use in your redesign: a prism, cone, pyramid, or some combination of these shapes (or any three-dimensional geometric figure you can describe).*
2. *Determine the dimensions. How much material will this design need?*
3. *Explore how companies are approaching eco-friendly products. What did you learn? Why might this be important?*
4. *What do you want to aspire to as you reflect on the design of your product?*

Invite students to revisit and redesign their item.

Part 2: Design a Package

Your job is to design an eco-friendly package for shipping the controller you designed.

1. *Explore how other products are currently packaged. How eco-friendly are the current designs? How can the current design be improved upon? What elements in your analysis would make the packaging eco-friendlier?*
2. *Design an eco-friendly way to package the controller you imagined in Part 1. Make note of the packaging shape and dimensions.*
3. *Prepare a presentation that illustrates the design of the controller and packaging. Within your presentation:*
    a. *prove the controller fits.*
    b. *share how your design is eco-friendlier than what is currently used.*
    c. *share the benefits and drawbacks of packaging items like your design compared to the ones we see in stores.*

Sample Learning Experience

Invite students to share hobbies and interests they have outside of school. In this task students have the opportunity to explore some of the decisions a small business owner and designer has to make. Part 1 offers an opportunity for students to explore and design a new controller. They may use a collection of materials, card stock, pre-designed cardboard cylinders, and paper as well as computer-aided design software to develop their controller design. Give students opportunities to research and brainstorm product designs, with at least one cycle of feedback.

Here are some resources for students to explore:

Ecofriendly Shipping Supplies
What Does Eco-Friendly Actually Mean?
What Does Eco-Friendly Mean?
Xbox One's Production Cost Revealed
24 Eco Friendly Packaging Examples that Benefit your Brand
18 Creative Sustainable Packaging Design Examples

Here are some questions to facilitate a feedback discussion:

- *What do you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *Do you think this design is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

If not raised by students, consider inviting them to think about the ramifications of mass producing a design. Give students an opportunity to redesign their product based on insights gained from feedback sessions.

Part 2 of this task offers students an opportunity to explore current packaging practices and reflect on eco-friendly alternatives.

Offer them the opportunity to work in pairs or small groups to complete the task. Give the option of using chart paper, Google Slides, or other creative outlets to showcase their work. Give students the opportunity to return to their work to revise, as necessary.

In this example, students are:

- engaging with cognitively demanding tasks in heterogeneous settings. Students engage in iterative cycles using geometric concepts and methods to design a product and its packaging (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- building understanding through application of geometric concepts of surface area and volume of two- and three-dimensional aspects (LP 3).

- able to explore areas of interests and develop a lens for aspirations, as they explore the world of product design and packaging (LP 4).
- connecting relevant learning to authentic situations (LP 6).
- able to use technology as a tool, as they employ computer-aided design software to perform the design of their product and packaging (LP 7).

Example 2

Students are given a problem like this:

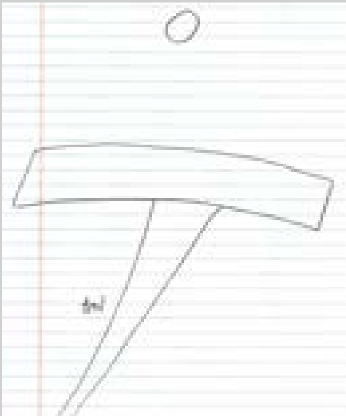
Muir Woods is a National Park in the Bay Area of California known for its distinctively tall Redwood Trees. You are visiting California with a friend who's a different height than you. You both decide to go for a hike on a trail that doesn't have a map. Off into the distance you see a break in the trees, and you are trying to figure out how long this hike will be because you're both ready to take a break.

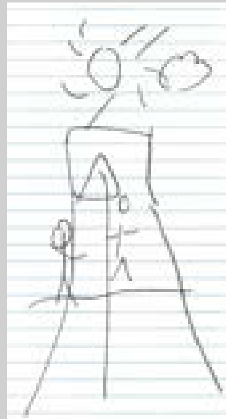
1. *How far can you see into the distance compared to your friend?*
2. *If the break in the trees is where your eye meets the horizon, how far away is the break?*
3. *Together you can walk a mile in 16 minutes. How long will it take both of you to get to the break in the trees?*

(Adapted from Illustrative Mathematics Task How Far is the Horizon?, 2016)

Sample Learning Experience

Begin by inviting students to sketch what this scenario may look like. Select some sketches to highlight and reflect on. A couple of preliminary sketches may look like this:



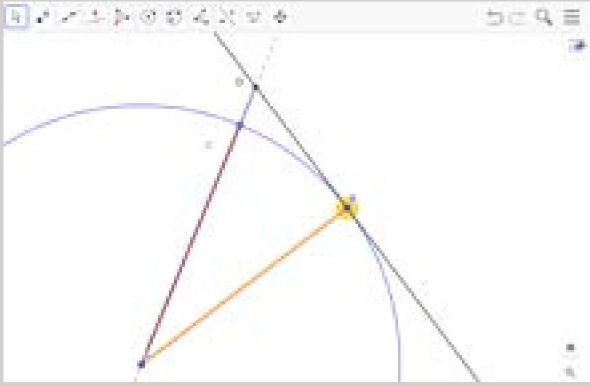


As students share their sketches with classmates, invite them to make sense of each other's sketches. Consider the following questions:

- *What are some things you appreciate about this sketch?*
- *What questions does this sketch surface for you?*
- *What about these sketches is helpful in giving us a way to use the mathematics we know to answer the questions at hand?*
- *What can you take from this approach to apply to your own work?*
- *What are some quantities we could track and work to represent in mathematical ways? What questions could we answer with that information?*

Give students time to make note of insights they have gained from this discussion and to revise their preliminary sketches. Monitor and select additional student work to discuss, using applicable questions from above.

As student groups work, monitor and note different assumptions and approaches they take to define a representation and quantities that model this scenario. Select varied approaches to share with the whole class. The image below is one where a student has a zoomed-out side-view of the situation.



Consider using the following prompts to facilitate the conversation:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, give students an opportunity to reflect on and revise their work.

In this example, students are:

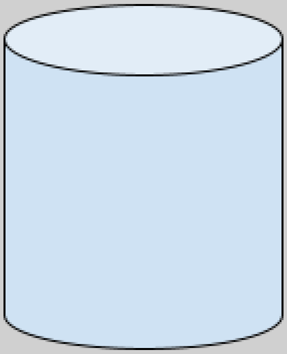
- engaging with a cognitively demanding task. It offers little in the way of traditional diagrams or measurements that support routine computations. Instead, students must visualize or create their own diagram, including making assumptions or researching information about the radius of the Earth (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- building conceptual understanding through reasoning. Students must construct a series of logical arguments and assumptions in their modeling as they use circles and right triangles to model this situation (LP 3).

Example 3

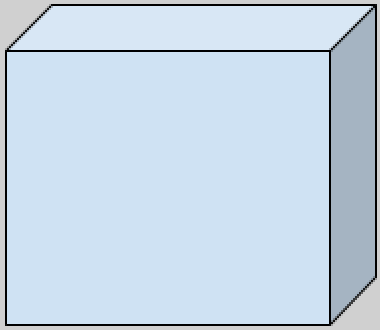
Students are given a problem like this:

What are some ways these two physical representations of solids are alike? different?

A



B



Now consider the following questions.

1. *What information would you need to calculate the volume of each solid?*
2. *What is the same and different about how you would find the volume of each solid?*

These images show the same number of coins arranged in different ways:

C

D

3. *How are the two coin stacks different from each other?*
4. *Does either stack resemble a geometric solid? If so, which stack and solid?*
5. *How do the heights of the two stacks compare?*

6. *How do the volumes of the two stacks compare? Explain your reasoning.*

(Adapted from Cylinder Volumes and Cross Sections and Volume by Illustrative Mathematics, 2019)

Sample Learning Experience

Students are asked to look at the solids depicted in A and B, and to answer the questions that follow independently. Students then collaborate with a partner while a teacher observes and selects student responses to share with the whole class for discussion, such as these for Question 2:

- "The figures are the same because they are both 3-dimensional."
- "The figures are the same because they both have a height."
- "The figures are the same because they will both have the same volume."
- "The figures are the same because you can calculate the volume the same way for both."
- "The figures are the same because you can find the volume by multiplying the area of the base times the height."

As these responses are selected and shared, the teacher may pose discussion questions like these:

- *"Do you agree with this statement? Why or why not?"*
- *"What adjustments might you make to the statement to make it more accurate?"*

Students are then given time to examine figures C and D and respond to the questions independently before conferring with a partner. Time could then be focused on Question 4; the teacher could ask students to choose whether they think the volumes are the same or different, and facilitate a debate where students offer logical arguments one way or the other to convince their classmates.

In this example, students are:

- given a non-routine task. Instead of being given figures or models labeled with specific dimensions (e.g., "the height of the cylinder is 30 cm"), students must make assumptions and think conceptually about the meaning of volume (LP 1).
- building a conceptual understanding through reasoning. Students' conceptual understanding is strengthened by comparing figures. Instead of computing, they must use the definition of volume to think about missing information, form comparisons, and explain their thinking (LP 2).
- engaging in social activities as they share their reasoning with a partner, discuss each other's ideas as a class, and debate a claim about volume (LP 3).

Example 4

Students are given a scenario like this:

The US Congress is made up of two branches—the US Senate and the House of Representatives. The US Senate is made up of two senators from each state (voted/appointed every six years). Each state has appointees for *the House of Representatives dependent upon the state's population (voted/appointed every two years). The information gathered to determine a state's population, and thus its number of representative seats, comes from the US Census, a demographic survey conducted every 10 years across the country.*

1. *If the number of seats a state holds in the House of Representatives is determined by the census data, explain how population/demographic data can accurately/inaccurately impact Congress.*
2. *There are only 13 states that have over 10 seats in the House of Representatives. The other 37 have less than 10.*
    a. *Make an educated guess naming five of the 13 states that have over 10 seats. Explain your reasoning.*
3. *There are seven states that have only one seat in the House of Representatives.*
    a. *Make an educated guess naming three of the seven states with only one seat. Explain your reasoning.*
4. *Some people argue that having the House of Representatives is fair because it allows more densely populated states to have more votes for their people. Population density measures the number of people per square mile of land in any state.*
    a. *Find the three states with the highest number of seats in the House of Representatives. Explain why they were or were not in your original guess above.*
    b. *Calculate their population density and explain what the number means in context.*
    c. *Find three states with only one seat in the House of Representatives. Explain why they were or were not in your original guess above.*
    d. *Calculate their population density and explain what the number means in context.*
    e. *Based on the population density, do you think the decision-making in the House of Representatives is fair? Why or why not?*
5. ***Extension**: Find the number of representatives for the state you live in. Then, calculate your state's population density.*
    a. *Use the data you have collected to compare your state's representation to that of the three with the most representative seats. Does your state have an adequate number of seats? Explain why or why not.*

(Information Sources From: How Your State Gets Its Seats, United States House of Representatives Seats by State)

Sample Learning Experience

Begin by giving students time to explore the resources How Your State Gets its Seats and United States House of Representatives Seats by State as they build an understanding of this context to respond to the first three questions. Structure time so that students can move between independent and small group work as they explore these resources. Give them access to tools like spreadsheets to organize their work and expedite computations.

As students work, monitor for varied approaches to these prompts. Invite selected groups or individuals to share their thinking with the class. Consider using the following prompts to facilitate the conversation:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Monitor and select student work on Question 4. Ask selected students or groups to prepare a presentation on their work. Orchestrate a whole group discussion around the varied ways students approached this work. Use applicable questions from above. Following the discussion, give students an opportunity to reflect on and revise their work.

Time permitting, give students time to work on the extension, Question 5. Invite them to prepare a presentation using tools of their choice: video, slideshow, posters, website, etc.

In this example, students are:

- engaging with cognitively demanding tasks as they make sense of the United States Congress, and more specifically, the allotment of the number of elected members of the House of Representatives per state (LP 1).
- set up to have social interactions, as they share and discuss their work with each other. These interactions may happen live in a physical classroom or asynchronously in a remote setting (LP 2).
- building conceptual understanding through reasoning. As students make sense of population density and representation in the US House of Representatives, they are continuously reasoning abstractly and quantitatively (LP 3).
- able to explore areas of interests and develop a lens for analyzing existing structures, as they make sense of the mathematics behind the US House of Representatives (LP 4).
- seeing the relevance of this real-world situation as they reflect on the implications of their analysis (LP 6).
- able to use technology as a tool, as they employ Google Sheets to perform their analysis (LP 7).

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M152 Reasoning and Proof through Congruence

Badge Catalog Description

Congruence provides fertile ground for reasoning about geometric figures, allowing us to hone our deductive skills for use in a wide variety of contexts. We can reason with rigid motions to develop criteria for congruent triangles, enabling us to then prove a wide array of theorems, as well as find unknown angle measures and segment lengths in a variety of figures. Specifically, we can use available evidence to argue that two triangles are congruent or that two angles have the same measures.

In M152 Geometry: Reasoning and Proof through Congruence, you will construct and critique logical arguments as you explore relationships between geometric figures. You will perform geometric constructions and describe a sequence of rigid motions to prove that one figure is congruent with another, using technology to aid in performing constructions and transformations. Acquiring the skills of geometric reasoning in these areas leads to the understanding of analytical approaches to geometry and opportunities to apply logical reasoning in fields beyond mathematics. Congruence is useful for careers in a variety of fields, like mathematics, graphic design, animation, art, and physics.

Suggested prerequisites for this badge: basic understanding of angles, triangles, and quadrilaterals.

The M152 Reasoning and Proof through Congruence badge focuses on students' construction of logical arguments within the context of the geometric concept of congruence. Students develop and revise a variety of arguments, ranging from simple statements of justification to complex proofs, while learning about congruence through rigid transformations. "Congruence is defined in terms of rigid motions—reflections, rotations, and translations" (Common Core Standards Writing Team, 2013, p.13).

According to *Catalyzing Change in High School Mathematics*, "The four congruence transformation types are intuitive, in that figures can be seen as a whole rather than just studied for their angles and sides. This provides a visual foundation for the identification of corresponding parts of figures related by a transformation" (National Council of the Teachers of Mathematics, 2017, p. 70). The badge can be completed prior to M153, concurrently with M153, or after M153. In the case that it is completed after M153, it is important to recognize students will begin with a strong intuitive understanding of congruence by way of transformations, positioning students to utilize formal definitions of rigid transformations.

As students engage in M152, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M152 badge.

M152 Content and Practice Expectations

152.a	Use definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
152.b	Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using graph paper, tracing paper, geometry software, etc.
152.c	Specify a sequence of transformations that will carry a given figure onto another.
152.d	Given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
152.e	Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
152.f	Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.
152.g	Prove theorems about lines and angles.
152.h	Prove theorems about triangles.
152.i	Prove theorems about parallelograms.
152.j	Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Learning Principles

In M152 Reasoning and Proof through Congruence, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M152

A key component of M152 is an emphasis on rigid motions. Whereas a typical instructional unit might focus on performing transformations on coordinates without connections to proof and congruence, in M152, students should:

- be given opportunities to explore and discuss rigid transformations using technology and physical models, starting with cases that do not involve coordinates (LP 7, LP 2).
- explore and reason about the properties of rigid motions (LP 3).

- make and justify claims about the effect of transformations on points, line segments, angles, and other figures (LP 3).
- use rigid motions to logically develop the triangle congruence criteria (SSS, SAS, SAA) (LP 3).

Often when engaging with the ideas of proofs and congruence, students are asked to spend time on lower-level skills, such as the memorization of triangle congruence criteria. When proofs are introduced, attention is paid to the notational conventions of two-column proofs. Time is also often allocated to finding missing measurements in diagrams, absent justification, or reasoning. In M152, students should instead develop triangle congruence criteria from a definition of congruence based on rigid motions and focus deeply on the development of logical arguments. Students should:

- spend significant time constructing a variety of logical arguments and proofs, which can take a variety of forms (prose, diagrams, two-column, etc.) (LP 3).
- use repeated reasoning to make generalizations about the transformations (LP 3).
- engage with their peers to describe their thinking using precise language and critiques and revise their logical arguments (LP 2).
- see geometry and the development of proofs as a form of problem-solving and engage in frequent experiences where they must make sense of given information, construct diagrams, test solution methods, and revise their thinking (LP 1).
- engage in learning experiences that emphasize reasoning over notation and other conventions (LP 2).
- employ a variety of tools, such as graph paper, tracing paper, and geometry software to illustrate and understand congruence (LP 7).

Evidence of Learning

In M152, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts to present evidence of their learning related to the badge content and practice expectations. Students submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
152.a: Use definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.	i. explain what a rotation, reflection, or translation is using appropriate terms: angle, circle, perpendicular line, parallel line, or line segment.
152.b: Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using tools such as graph paper, tracing paper, or geometry software.	i. create a rotated, reflected, and/or translated figure using graph paper, tracing paper, or geometry software.
	ii. explain how you rotated, reflected, and/or translated a figure.
152.c: Specify a sequence of transformations that will carry a given figure onto another.	i. describe a sequence of transformations that would carry a figure onto another figure.
152.d: Given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.	i. write an argument or explanation for why two figures are congruent, using similarity transformations in your reasoning.
	ii. write an argument or explanation for why two figures are not congruent, using similarity transformations in your reasoning.
152.e: Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.	i. demonstrate using an example that if two triangles are congruent, then corresponding angles and corresponding sides are also congruent.
	ii. explain why, in general, if two triangles are congruent, then corresponding angles and corresponding sides are also congruent.
	iii. demonstrate using an example that if two triangles have corresponding angles and corresponding sides that are congruent, then the two triangles are also congruent.
	iv. explain why, in general, if two triangles have corresponding angles and corresponding sides that are congruent, then the two triangles are also congruent.
152.f: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.	i. demonstrate using an example that if two triangles meet one of the triangle congruence criteria (ASA, SAS, or SSS), then they are congruent according to the definition of congruence in terms of rigid motions.
	ii. explain why, in general, if two triangles meet one of the triangle congruence criteria (ASA, SAS, or SSS), then they are congruent according to the definition of congruence in terms of

Content and Practice Expectations	Indicators Choose an artifact where you...
	rigid motions.
152.g: Prove theorems about lines and angles.	i. find a missing length or angle measurement in a diagram containing lines and angles and explain how you determined your answer.
	ii. read a proof of a statement about lines and angles (a specific case or in general) and summarize or explain it in your own words.
	iii. write a proof of a statement about lines and angles (a specific case or in general), using rigid transformations or the concept of congruence in your reasoning.
152.h: Prove theorems about triangles.	i. find a missing length or angle measurement in a triangle and explain how you determined your answer.
	ii. read a proof of a statement about a triangle or triangles in general and summarize or explain it in other words.
	iii. write a proof of a statement about a triangle or triangles in general, using rigid transformations or the concept of congruence in your reasoning.
152.i: Prove theorems about parallelograms.	i. find a missing length or angle measurement in a parallelogram and explain how you determined your answer.
	ii. read a proof of a statement about a parallelogram or parallelograms in general and summarize or explain it in your own words.
	iii. write a proof of a statement about a parallelogram or parallelograms in general, using rigid transformations or the concept of congruence in your reasoning.
152.j: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).	i. construct a geometric figure using compass and straightedge, dynamic geometric software, or other tools.
	ii. explain how you constructed a geometric figure.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept

The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M152 Reasoning and Proof through Congruence (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

In the field of marketing, it is crucial to create a brand that stands out from the rest! Think about your favorite brand logos. What about those logos make them memorable?

One of the ways that companies capture their audience is through symmetry. Symmetrical patterns are said to give the human brain a sense of familiarity and order, making it easier to remember products. Take a moment to explore the article Why Do We Get So Much Pleasure From Symmetry? *from the website howstuffworks to learn more.*

1. *Name a brand that comes to mind for you initially. Explain any symmetry you see represented in the brand's logo.*
2. *Select two company logos with symmetry, one with line symmetry and one with rotational.*
    a. *Make an argument for which logo is more aesthetically pleasing in terms of the type of symmetry.*
    b. *Work to give detailed explanations wherever possible. For example, for the logo with rotational symmetry, identify and explain the degree of the rotation.*
3. *Let's explore a company logo that took a different approach by not using symmetry in its design.*
    a. *Find and select a logo you like that fits this description. What is something about this logo you appreciate?*
    b. *Redesign the logo so it is visually appealing **and** has rotational symmetry.*
    c. *Explain your chosen degree of rotation for your new logo. What is the degree or rotation you decided to use? In what ways does this honor what you appreciate about the logo or company?*

Sample Learning Experience

Launch this lesson with a "Which one doesn't belong?" warm up activity. Create a four-image comparison of logos that are a mix of symmetrical and non-symmetrical. Give students silent thinking time to create an argument around their choice. Have them share out in groups or to the whole class. Consider using Four

Corners to have students share their arguments with others who selected the same choice before sharing with the whole class.

Questions that can be posed during share out:

- *Which logos have line symmetry?*
- *Which logos have rotational symmetry? What do you think is the degree of rotation? What did you pay attention to in the logo to land on that response?*
- *What other logos come to mind? Which of these have symmetry? How can you tell?*

Give students time to research logos as they work to respond to Question 2. Invite them to research logos in order to identify two logos: one with line symmetry and another with rotational. Monitor their progress and select student work that illustrates understanding of symmetry. Have students share out and explain identified symmetry for each logo selected. Encourage them to ask questions to each other. Consider using the following questions:

- *What is clear to you from this explanation?*
- *What is something you want to hear more about?*
- *What additional questions surface for you?*

Following the share out, give students an opportunity to further develop, modify, or revise their work.

Give students time to research logos as they work on Question 3 of the task. Invite them to identify a logo that does not utilize symmetry in its design. Give students time to begin exploration independently, then invite them to work in pairs to engage in the design process by selecting one of their logos to work together to make it more aesthetically pleasing. Give students the option to utilize tools and web-based applications, like tracing paper, Desmos, GeoGebra, Canva, or Photoshop as they redesign the logo of their choice.

Consider concluding this exploration by having students participate in a gallery walk to showcase their designs.

In Example 1, students are:

- given a cognitively demanding task requiring them to reason about transformations. The learning experience is structured to give all students access by inviting them to select which one does not belong and builds up to engage every student in the redesign of a logo using their understanding of transformations (LP 1).
- working individually and collaboratively in pairs to complete the task, allowing for social interaction among students. They are also encouraged to ask questions to each other to better understand the thinking process. (LP 2).
- building conceptual understanding through reasoning, as they identify symmetry in logo designs and create a logo with rotational symmetry (LP 3).
- given space to develop agency, as they choose their solution pathways and reflect on how their approach might change based on engagement with their peers' work. Students also reflect on how the designs they create reflect what they value (LP 4).
- given insight into how mathematics is used in the field of marketing by understanding how symmetry can enhance logo designs (LP 6).

- using technology as a tool, as they employ tracing paper and/or web-based apps to identify the degree of rotation and redesign a logo (LP 7).

Example 2

Students are given a problem like this:

Choose two distinct points A and B in the plane.

A. *For which points C is △ABC a right triangle?*
B. *For which points C is △ABC an obtuse triangle (that is, a triangle with one obtuse angle)?*
C. *For which points C is △ABC an acute triangle (that is, a triangle with three acute angles)?*

Justify your responses.

(Classifying Triangles, Illustrative Mathematics, 2016)

Sample Learning Experience

Launch this task by giving students access to Geometry - GeoGebra or Desmos | Geometry in order to engage in active experimentation with the location of the third point. Students can explore different placements for the third point relative to the two given points and record observations they believe might be important as they work to respond to the prompt's three parts.

Monitor and select approaches that illustrate varied approaches and/or student-generated conjectures. Invite students to share and learn more about each other's approaches. Ask:

- *What do you appreciate about this approach?* Or *What about this conjecture makes sense to you?*
- *What questions does this approach surface for you?* Or *What are some questions we can ask about this conjecture? How might we work to construct an argument or explanation about what's happening?*
- *What can you take from this approach to apply to your own work?*

Provide additional time for students, working individually or with peers, to further develop, modify, or revise their work.

In Example 2, students are:

- given a cognitively demanding task that asks them to synthesize their learning about triangles. In this task, students construct geometric figures using compass and straightedge, dynamic geometric software, or other tools to fit a given condition (LP 1).
- working individually and collaboratively in pairs to complete the task, allowing for social interaction among students. They are also encouraged to ask questions of each other to better understand the thinking process (LP 2).

- building conceptual understanding through reasoning, as they work to construct and identify a third point of a triangle to meet a given constraint. In doing so, students have the opportunity to gradually formulate arguments and leverage both mathematical definitions and theorems in support of their mathematical ideas (LP 3).
- able to explore skills and dispositions as they work to solve a complex problem. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to create and explore their own conjectures (LP 4, 5, and 6).
- encouraged to use dynamic geometry software to facilitate exploration and construction of triangles as part of their active experimentation with the prompts (LP 7).

Example 3

Students are given a problem like this:

Part 1: Mathematics in Art

Artists like Kiayani "Kay" Douglas use mathematics as they design their artwork. In this task you will have the opportunity to learn more about Ms. Douglas and her art, as well as time and space to create your own artwork using some of the mathematical ideas she shares with us, and specifically tessellations.

Watch this interview with Ms. Douglas about her Portraits Collection and the use of tessellations: https://bit.ly/3Rh7tJM .

1. *What intrigued you about what Ms. Douglas shared? What questions did it surface for you?*
2. *What might you want to learn or explore further in preparation to create your own artwork?*
3. *What insights from Ms. Douglas' video do you want to incorporate into your artwork?*

Part 2: Create Your Own Artwork

In the video, we learned how Ms. Douglas designed the tessellations present in her portrait collection by working with a square. We also learned that working with tessellates and other tools, like ink and colored pencils, can be used to create images on that piece, much like Ms. Douglas' outline of Africa and the power fist.

For this task you will explore and design a tessellating piece for your final work of art.

1. *Select a polygon to base your template (tessellating piece) on: a square or other polygon (triangle, rectangle, or hexagon).*
2. *Explore ways to modify the polygon to create a piece that tessellates.*
3. *Iterate your piece to create a sample tessellation using your template. What transformations are in play? What images can you see surfacing? What might you draw on it to create the image you want to share?*
4. *Repeat Steps 1-3 until you find the shape you will use as the template of the tessellating piece in your artwork.*

Create your work of art. Consider how you will use color, space, and other elements to design artwork that you're proud of. Upon completing your artwork, write or record a statement addressing the following questions:

- *Tell us a little about yourself. What is something you learned about yourself in doing this work?*
- *Tell us about your art piece. What inspired this work? How would you describe the composition and elements of this piece? How would you describe what we're looking at?*
- *Tell us about your tessellating piece. How did you construct the template? How did you design it to take on the image(s) it currently holds? What transformations are evident in your template?*
- *Tell us about the tessellation. How does your tessellating piece fill the space? How did you use this space? How would you describe the transformations present in your tessellation?*

Sample Learning Experience

Part 1: Mathematics in Art

Begin by inviting students to share what they know about how mathematics may surface in art. Listen for students sharing elements such as shapes, symmetry, perspective, patterns, and proportion. Use these elements to set the stage for listening to Kiayani "Kay" Douglas' video. Share the video. Facilitate a discussion and have students reflect on what they heard.

Highlight the use of transformations to form the background for the shared artwork. Encourage students to visit Ms. Douglas' website, https://www.iamkaydouglas.com/, to explore and look at the images she highlighted more closely.

Move the conversation towards coming to a shared understanding of what a tessellation is and establish comfort with identifying the tessellating piece. Encourage students to identify and share where tessellations might be found in everyday life, e.g., tile work on floors and walls, some wallpaper patterns, and rugs.

Part 2: Create Your Own Artwork

Give students individual work time to develop a template for the tessellating piece. You may opt to use pattern blocks as a precursor to deepen an understanding of how polygons work to tessellate a plane. For example, we can tile using one polygon—tessellating piece—or we may use multiple pieces to tile a plane. Pattern blocks can make that visible. This exploration can lead to a list of polygons that students may want to consider as the basis of their tessellation.

Provide students multiple copies of various polygons they can use to explore different options, as illustrated here: A Simple Method For Creating Tessellations From Rectangles. As students work, encourage them to explore the impact of translating, reflecting, and rotating cut-outs on overall tessellation (Steps 2 and 3). Encourage students to work in pairs as they create their template to share and discuss their progress with each other. Make dynamic geometric applications, like Desmos and Geogebra, available for students to use to create their template and tessellation.

Monitor and select student work to share with the class. Ask selected students to share how they have approached the task of creating the template for the tessellating piece. Consider asking:

- *How did you go from the original polygon to the shape we see now?*
- *What transformations did you use?*
- *How would you describe what you see happening as you tessellate your piece? What transformations do you see in play?*

Encourage students to engage with and make sense of each other's designs, using these questions:

- *What do you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide opportunity for students to return to their work to revise, modify, and finalize their template.

Once they have finalized their template, consider having students display their template under a document camera or overhead, so that classmates can share what they might "see" in the template, e.g., a flower, a ship, a car, an animal, etc. This may help students with adding details to their template to create their artwork.

Provide the opportunity for students to return to their work to revise, modify, and finalize the design of their template.

Provide additional time for students to complete their artwork and determine a way to tell their story. You may opt to host a gallery walk when all pieces are complete and invite community members to attend.

In Example 3, students are:

- given a cognitively demanding task. They are asked to consider how transformations can be used to create a figure that tessellates a plane, create the template using one or more transformations, and create a tessellation using the template and adding details to bring meaning to the tessellation (LP 1).
- working individually and collaboratively in pairs and sharing their findings with the class as they progress through this exploration. Students are encouraged to ask questions to better understand the thinking process (LP 2).
- applying their understanding of transformations to create a figure that will tessellate and examining patterns that emerge from tessellating a plane (LP 3).
- bringing their identity to the fore as they interact with mathematical figures to create a personal piece of artwork (LP 4).
- given insight into how artists integrate mathematics to create meaningful and artistic designs (LP 6).

- encouraged to use dynamic geometry software to facilitate the exploration and construction of templates and tessellation (LP 7).

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M153 Reasoning and Proof through Similarity

Badge Catalog Description Similarity provides fertile ground for reasoning about geometric figures, allowing us to hone our deductive skills for use in a wide variety of contexts. We can build on reasoning with rigid motions to include dilations in order to develop criteria for similar triangles, enabling us to then prove a wide array of theorems that allow for determining unknown angle measures and segment lengths in a variety of figures, and for creating animated models using technology. In M153 Geometry: Reasoning and Proof through Similarity, you will construct and critique logical arguments as you explore relationships between geometric figures. You will perform geometric constructions and describe a sequence of transformations to prove that one figure is similar to another, using technology to aid in performing transformations. You will also test and apply theorems to reason about relationships between figures to prove similarity, as well as to construct visual representations of images your mind creates using segments, angles, triangles, or quadrilaterals. Acquiring the skills of geometric reasoning in these areas leads to the understanding of analytical approaches to geometry and opportunities for applying logical reasoning in fields beyond mathematics. Similarity is useful for careers in a variety of fields, like mathematics, graphic design, animation, art, and physics. Suggested prerequisites for this badge: basic understanding of angles, triangles, and quadrilaterals.

The M153 Reasoning and Proof through Similarity badge focuses on students' construction of logical arguments within the context of the geometric concept of similarity. Students develop and revise a variety of arguments, ranging from simple statements of justification to complex proofs, while learning about similarity through similarity transformations. "An advantage of the transformational approach to similarity is that it allows for a notion of similarity that extends to all figures rather than being restricted to figures composed of line segments" (Common Core Standards Writing Team, 2013, p. 17). The badge can be completed prior to M152 or concurrently with M152. In the case that it is done prior to M152, care must be taken to leverage students' intuitive understanding of congruence.

As students engage in M153 Reasoning and Proof through Similarity, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M153 badge.

M153 Content and Practice Expectations

153.a	Given a geometric figure and a dilation, draw the transformed figure using graph paper, tracing paper, geometry software, etc.
153.b	Specify a sequence of transformations that will carry a given figure onto a similar figure.

153.c	Verify experimentally the properties of dilations given by a center and a scale factor.
153.d	Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.
153.e	Using similarity transformations, explain the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
153.f	Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.
153.g	Use similarity to prove theorems about triangles.
153.h	Use similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Learning Principles

In M153, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting “doing mathematics,” but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M153

A key component of M153 is an emphasis on similarity transformations. Whereas a typical instructional unit might focus on performing transformations on coordinates without connections to proof and the concept of similarity, in M153, students should:

- be given opportunities to explore and discuss similarity transformations using technology and physical models, starting with cases that do not involve coordinates (LP 7, LP 2).
- explore and reason about the properties of dilations (LP 3).
- make and justify claims about the effect of dilations on points, line segments, angles, and other figures (LP 3).
- use similarity to logically develop the triangle similarity criteria (SSS, AA) (LP 3).

Often when engaging with the ideas of proofs and similarity, students are asked to spend time on lower-level skills, such as memorization of triangle similarity criteria. When proofs are introduced, attention is paid to the notational conventions of two-column proofs. Time is also often allocated to finding missing measurements in diagrams, absent justification, or reasoning. In M153, students should instead develop triangle similarity criteria from a definition of similarity based on transformations and focus deeply on the development of logical arguments. Students should:

- spend significant time constructing logical arguments and proofs, which can take a variety of forms (prose, diagrams, two-column, etc.) (LP 3).
- use repeated reasoning to make generalizations about similarity through transformations (LP 3).
- engage with their peers to describe their thinking using precise language and critique and revise their logical arguments (LP 2).
- see geometry and the development of proofs as a form of problem-solving and engage in frequent experiences where they must make sense of given information, construct diagrams, test solution methods, and revise their thinking (LP 1).
- engage in learning experiences that emphasize reasoning over notation and other conventions (LP 2).
- employ a variety of tools, such as graph paper, tracing paper, and geometry software to illustrate and understand congruence (LP 7).

Evidence of Learning

In M153, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts to present evidence of their learning related to the badge content and practice expectations.

Content and Practice Expectations	Indicators Choose an artifact where you...
153.a: Given a geometric figure and a dilation, draw the transformed figure using tools such as graph paper, tracing paper, or geometry software.	i. create a dilated figure using graph paper, tracing paper, or geometry software.
	ii. explain how you dilated a figure.
153.b: Specify a sequence of transformations that will carry a given figure onto a similar figure.	i. describe a sequence of transformations that would carry a figure onto a similar figure.
153.c: Verify experimentally the properties of dilations given by a	i. create a dilated figure using graph paper, tracing paper, or geometry software and describe the effect of dilations on line

Content and Practice Expectations	**Indicators** **Choose an artifact where you...**
center and a scale factor.	segments.
	ii. create a dilated figure using graph paper, tracing paper, or geometry software and describe the effect of dilations on angles.
153.d: Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.	i. write an argument or explanation why two figures are similar, using similarity transformations in your reasoning.
	ii. write an argument or explanation why two figures are not similar, using similarity transformations in your reasoning.
	iii. use the definition of dilation within your argument.
153.e: Using similarity transformations, explain the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.	i. demonstrate, using similarity transformations an example, that similar figures have corresponding angles that are equal in measure.
	ii. explain why, in general, similar figures have corresponding angles that are equal in measure, using similarity transformations in your reasoning.
	iii. demonstrate, using an example, that similar figures have corresponding sides that are proportional.
	iv. explain why, in general, similar figures have corresponding sides that are proportional, using similarity transformations in your reasoning.
153.f: Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.	i. demonstrate, using an example, that if two figures have two corresponding angles that are equal in measure, then they are similar.
	ii. explain why, in general, two figures that have two corresponding angles that are equal in measure are similar, using similarity transformations in your reasoning.
153.g: Use similarity to prove theorems about triangles.	i. read a proof of a statement about a triangle or triangles in general and summarize or explain it in your own words.
	ii. write a proof of a statement about a triangle or triangles in general, using similarity transformations or similarity criteria in your reasoning.
153.h: Use similarity criteria for triangles to solve problems and to prove relationships in geometric	i. find an unknown length measurement in a triangle and explain how you determined your answer, using similarity transformations or similarity criteria in your reasoning.

Content and Practice Expectations	Indicators Choose an artifact where you...
figures.	ii. find an unknown angle measurement in a triangle and explain how you determined your answer, using similarity transformations or similarity criteria in your reasoning.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

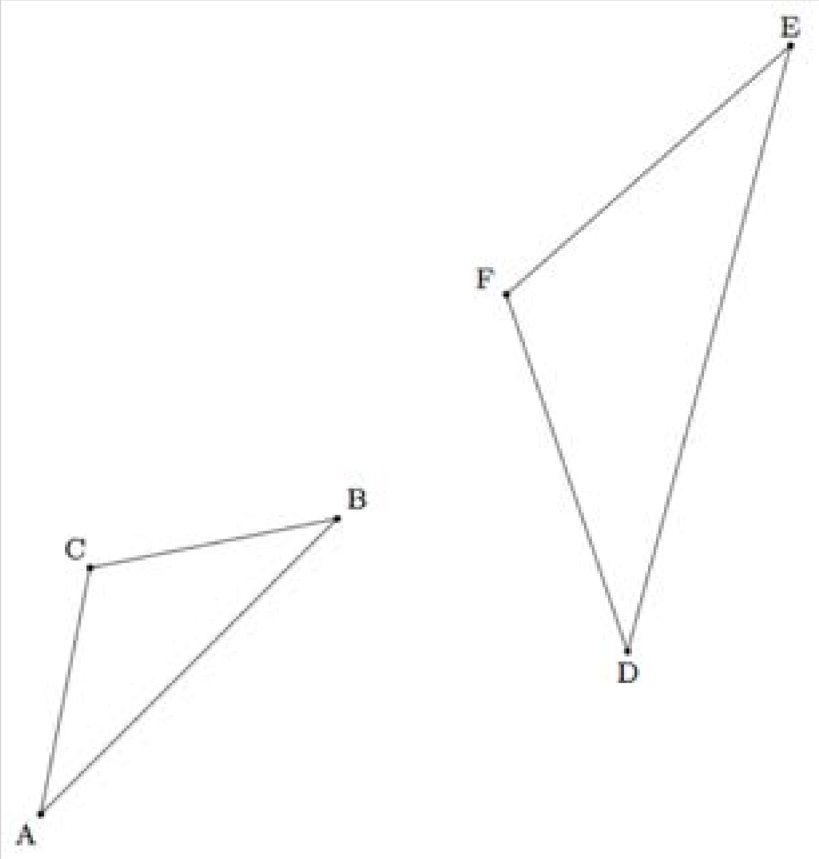
Annotated Examples M 153 Reasoning and Proof through Similarity (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

In the two triangles pictured below, $m(\angle A) \cong m(\angle D)$ *and* $m(\angle B) \cong m(\angle E)$ *):*



Using a sequence of translations, rotations, reflections, and dilations, show that △ABC is similar to △DEF.

(Illustrative Math task Similar Triangles, 2016)

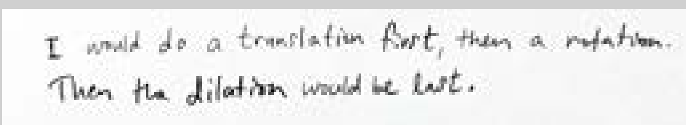
Sample Learning Experience

Begin by giving students quiet time to make sense of the information given and process the task they have to complete. Provide them with physical tools, such as transparencies or patty paper, or digital geometry software to experiment with different transformations.

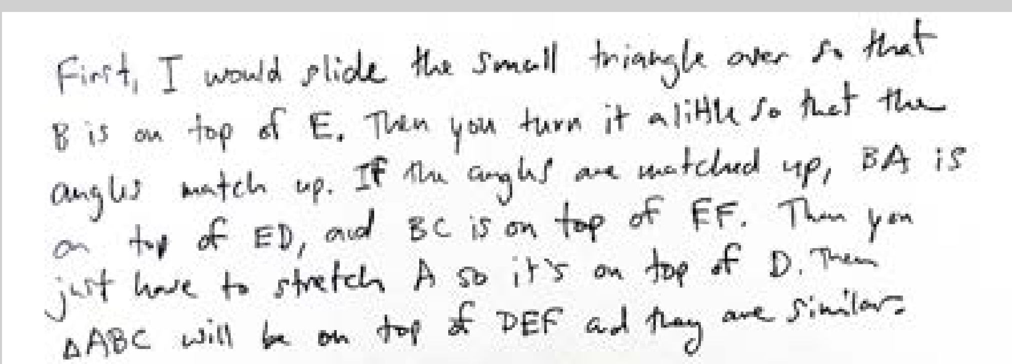
After some individual time, encourage students to discuss progress with a partner or small group. Encourage students to make their thinking visible. Initially, focus a conversation on how students interpreted and used the given information. Consider pulling the class together to discuss and share students' labels and use of tools to build shared understanding of the known information about the triangles. Monitor and select student work that aids in the conversation. Note some may mark up their paper whereas others may use their transparency or tracing paper to arrive at the conclusion that the given corresponding angles are congruent.

Monitor for, select, and share student work that embodies different approaches and levels of formality for discussion. Listen for the various ways students name the sequence of transformations. Note and select various levels of formality students are comfortable with as they share their work with each other. For example:

Sample Student Response 1

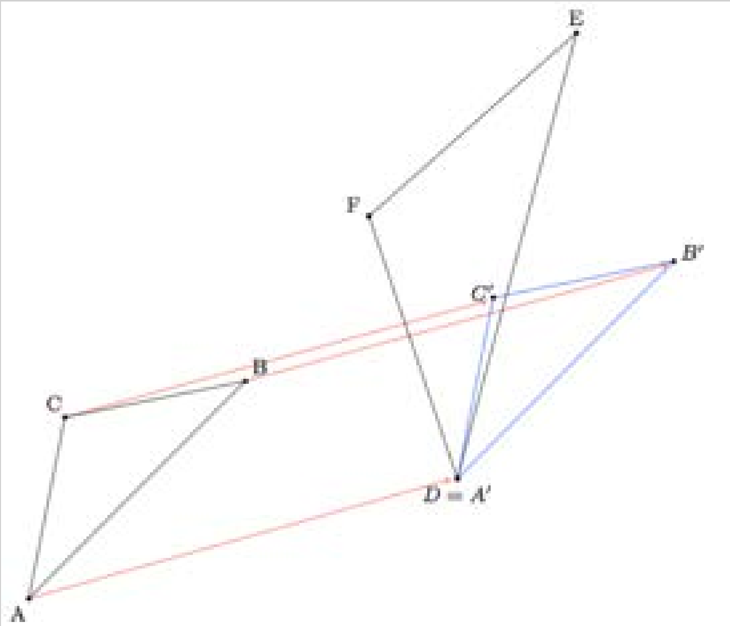

I would do a translation first, then a rotation.
Then the dilation would be last.

Sample Student Response 2

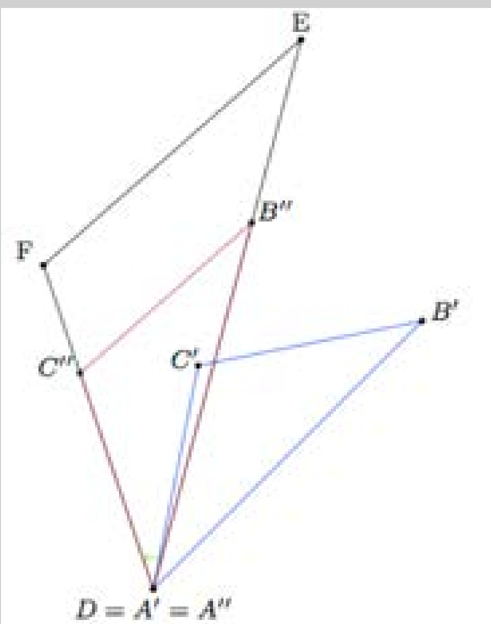

First, I would slide the small triangle over so that
B is on top of E. Then you turn it a little so that the
angles match up. If the angles are matched up, BA is
on top of ED, and BC is on top of EF. Then you
just have to stretch A so it's on top of D. Then
ΔABC will be on top of DEF and they are similar.

Sample Student Response 3

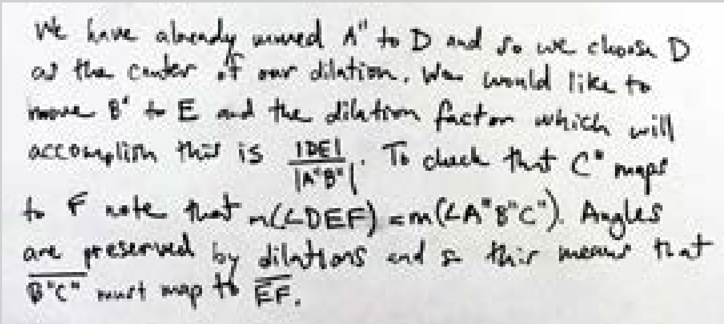

Translate A to D as shown below:



Then rotate $\triangle A'B'C'$ by angle $C'A'F'$ as shown below:



Note that $\overrightarrow{A''C''} = \overrightarrow{AC}$ because of the angle of rotation. Note too that $\overrightarrow{A''B''} = \overrightarrow{DE}$; this is true because $m(\angle B''A''C'') = m(\angle BAC)$, since rigid motions preserve angle measures, and $m(\angle BAC) = m(\angle EDF)$ by hypothesis.


We have already moved A'' to D and so we choose D as the center of our dilation. We would like to move B'' to E and the dilation factor which will accomplish this is $\frac{|DE|}{|A''B''|}$. To check that C'' maps to F note that $m(\angle DEF) = m(\angle A''B''C'')$. Angles are preserved by dilations and so this means that $\overline{B''C''}$ must map to $\overline{EF}$.

As students share their sketches and reasoning with classmates, invite them to make sense of each other's sketches. Consider the following questions:

- *What are some things you appreciate about this explanation?*
- *What questions does this explanation surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work and share with each other again. Focus this second discussion on changes or adjustments that they have made based on their peers' work.

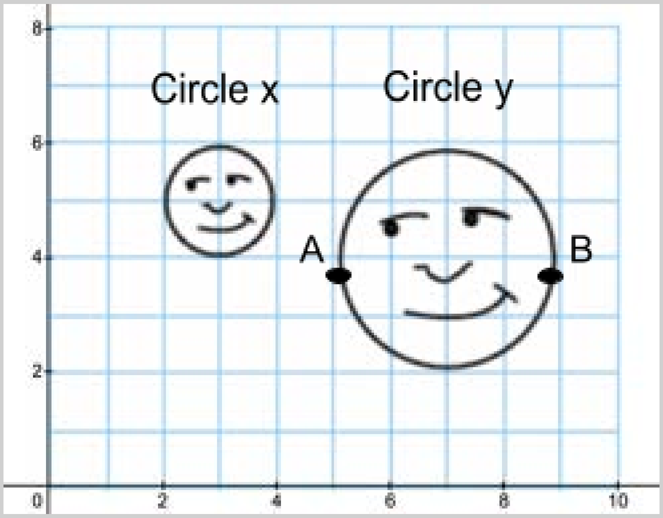
In this example, students are:

- given a cognitively demanding task. It requires students to reason using transformations that two triangles are similar. The learning experience is structured to give all students access by inviting them to write explanations with varying degrees of formality and giving an opportunity for revision after engaging with some of their peers' explanations (LP 1).
- working collaboratively through structured opportunities to ask questions to each other to better understand the thinking process (LP 2).
- building conceptual understanding through reasoning, as they must construct a logical argument based on their understanding of transformations and the given triangles (LP 3).
- encouraged to have agency, as they choose their solution pathways and are given space to reflect on how their approach might change based on engagement with their peers' work (LP 4).
- given an opportunity to use geometry software to experiment with transformations (LP 7).

Example 2

Students are presented with a problem like this:

The picture below depicts a dilation between Circle x and Circle y. A dilation is defined as, "A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor" (Mathematics Glossary CCSS).



1. *Given the position of the images, is Circle y a dilation from inside, on, or outside Circle x?*

2. *Determine the scale factor of the dilation to produce Circle y.*

3. *Use Geogebra to recreate the image online. Then, find the center of dilation from Circle x to Circle y.*

4. *Given a scale factor of 4, what is the set of coordinates of the image of points A and B? What was the center of the dilation that resulted in those coordinates? Would a different center of dilation result in the same coordinates you found? Why or why not?*

5. *Given a scale factor of 1/4, explain how Circle y would change. Under what conditions would this happen?*

6. *Revisit the definition of a dilation: "A transformation that moves each point along the ray through the point emanating from a fixed center and multiplies distances from the center by a common scale factor." How did you use parts of the definition to respond to the questions above?*

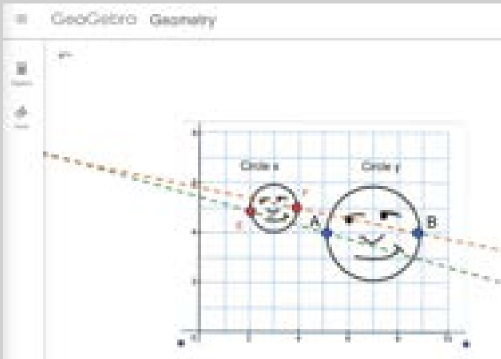
Sample Learning Experience

Introduce the lesson by displaying the given image. Ask students: "*What do you notice? What do you wonder?*" Chart student responses close to the image, in a place that is visible to the whole class. Encourage them to elaborate in order to make ways they are attending to structure visible to themselves and their classmates.

Introduce the definition provided and ask students to make connections between the definition and what they shared on what they noticed. Ask: "*What connections do you see between what you noticed and this definition?"* Give students a minute of quiet time to think, then share with a partner before eliciting responses from the whole group. Chart student connections close the previous collection of student-generated ideas.

If the concept of dilations is new to students, consider using excerpts from *Illustrative Mathematics Grade 8 Unit 2 Lesson 2* as a way to introduce it before engaging in this task.

Give students time to engage in Question 1 individually, followed by partner or small group work. Monitor and select varied approaches to this prompt to share for whole class discussion.

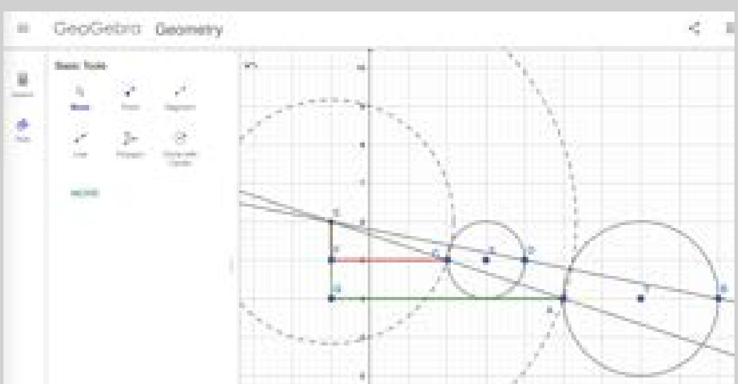


As students share their approach with classmates, invite them to make sense of each other’s work. Consider the following questions:

- *What are some things you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again. Focus this second discussion on changes or adjustments they have made based on their peers’ work.

As students return and work on Questions 2 and 3, monitor and select varied approaches to these prompts for whole class discussion.



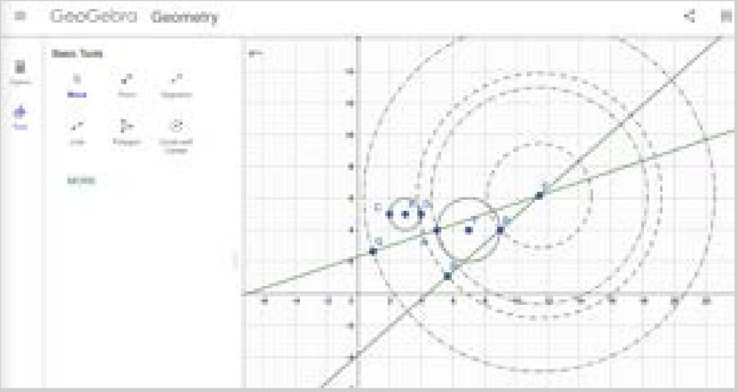
As they share their approach with classmates, invite students to make sense of each other’s work. Consider the following questions:

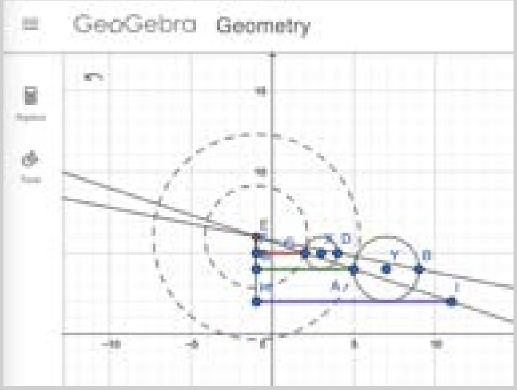
- *What are some things you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again.

As students return and work on Questions 4 and 5, monitor and select varied approaches to these prompts for whole class discussion. Focus this discussion on the role the center of dilation plays in determining where dilated images live on the plane.

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again.





For Question 6, consider using the math language routine, Stronger and Clear to give students a structure with which to synthesize their learning. Begin by posing the prompt and giving students time to write their preliminary response individually. Then give students three to four minutes to share their response with a peer. Then give students a moment to write a note to themselves based on what they heard from their classmate. Give students another opportunity to pair up with a different classmate and share responses again. Finally, give students time to revise their preliminary response.

In this example, students are:

- given a cognitively demanding task. They are asked to consider how the scale factor of a dilation creates a larger or smaller image. Students also have opportunities to grapple with and understand the role the center of dilation plays in this transformation (LP 1).
- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions of each other to better understand the thinking process, as well as given space to learn from their peers (LP 2).

- negotiating the meaning of dilations as they move back and forth between the definition and the actions they take in manipulating a given image (LP3).
- given space to develop agency, as they choose their solution pathways and reflect on how their approach might change based on engagement with their peers' work (LP 4).
- using technology as a tool, as they employ straight edges and/or web-based apps to identify the elements that define a given dilation, as well as explore alternate possibilities (LP 7).

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M154 Coordinate Geometry

Badge Catalog Description
How do pilots keep their aircraft on course? How are algebra and geometry related? Coordinate Geometry, also known as analytic geometry, establishes a connection between geometric curves and algebraic equations. In M154 Coordinate Geometry, you will reason and use algebra to do geometry, and use geometry to do algebra. As you learn coordinate geometry, you will investigate and make sense of conic sections, produce equations and graphs for circles, ellipses, and hyperbolas, and solve simple systems algebraically and with technology systems consisting of linear and quadratic equations. Using coordinates to answer questions about shapes in the coordinate plane will strengthen your ability to compute perimeters of polygons and areas of triangles and rectangles and to determine the equation of a line parallel or perpendicular to a given line. You will strategically use technology to explore the graphs of geometric curves and the related algebraic equations. For example, when designing an architectural arch, you might use graphing technology to manipulate an equation and see the result. Coordinate Geometry is useful for careers in a variety of fields, like computer graphics in games and films, space science, aviation, and engineering. Suggested prerequisites for this badge: basic understanding of angles, triangles, quadrilaterals, and circles; M101 Linear Equations: Concepts and Skills. This badge is suggested as a prerequisite for: Precalculus and Calculus.

The M154 Coordinate Geometry badge puts mathematical reasoning at the center of how students engage with algebra while using geometric ideas. Students develop logical reasoning, conceptual understanding, and problem-solving skills as they engage with coordinate geometry. As they use the coordinate plane to connect geometry and algebra, students will deepen their understanding of geometric concepts. According to the High School Geometry progression, students are expected "to express geometric properties with equations and use coordinates to prove geometric theorems algebraically" (Common Core State Standards Writing Team, 2016, p.5).

As students engage in M154 Coordinate Geometry, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M154 badge.

M154 Content and Practice Expectations

154.a	Translate between the geometric description and the equation for a conic section, specifically of a circle and parabola.

154.b	Use coordinates to prove simple geometric theorems algebraically.
154.c	Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.
154.d	Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
154.e	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Learning Principles

In M154 Coordinate Geometry, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M154

Whereas a typical instructional unit on coordinate geometry might begin with a focus on plotting points, memorizing formulas, or proving theorems disconnected from real-world contexts, in M154, students should:

- use formulas and equations to interpret the meaning of geometric concepts, derive equations, and prove geometric ideas (LP 3).
- apply rules, look for patterns, and analyze structure to prove properties of geometric figures (LP 3).
- regularly encounter real-world tasks that connect geometry and algebra (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- collaboratively explore how geometry and algebra are connected, share solution methods, and make thinking visible (LP 2).

In M154, students should:

- engage with tasks to discover geometric concepts by applying algebraic formulas to geometric figures and definitions. For example, students should:
    - reason about and understand how to derive the equations of circles and parabolas.
    - make connections between the different representations of the graphs of conic sections.
    - verify algebraically the properties of geometric figures by understanding the importance of and applying the formulas associated with the coordinate plane: slope formula, distance formula, midpoint formula, and section formula (partitioning a line segment).
    - use coordinates to verify geometric relationships by using the Pythagorean theorem, area, and perimeter.
    - reason about the relationships between the algebraic and geometric definitions.
    - justify algebraically the relationships between the slopes of parallel and perpendicular lines and geometric figures on the coordinate plane (LP 3).

- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that coordinate geometry can authentically be applied in real-world situations (LP 6).

Often, coursework with coordinate geometry is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use graphing software to allow for focus on understanding, seeing patterns, forming generalizations, and testing conclusions (LP 7).
- use various technologies, like interactive sketches (LP 7).

Evidence of Learning

In M154 Coordinate Geometry, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations. Students submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
154.a: Translate between the geometric description and the equation for a conic section, specifically of a circle and parabola.	i. explain how the Pythagorean Theorem can be used to derive the equation of a circle.
	ii. write the equation for a parabola using the focus and directrix.
154.b: Use coordinates to prove simple geometric theorems algebraically.	i. use the distance formula to find the distance between two points on a coordinate plane.

Content and Practice Expectations	Indicators Choose an artifact where you...
	ii. use coordinates along with the slope formula, distance formula, and/or midpoint formula to prove simple geometric theorems algebraically.
	iii. use coordinates to prove that a quadrilateral is, or is not, a specific quadrilateral
154.c: Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.	i. classify lines or segments as parallel, perpendicular, or neither.
	ii. write the equation of a line parallel or perpendicular to a given line and passing through a given point.
	iii. solve problems involving equations of parallel and perpendicular lines.
154.d: Find the point on a directed line segment between two given points that partitions the segment in a given ratio.	i. use the midpoint formula, the section formula, and the distance formula to find the partition point of a given directed line segment.
	ii. use dilations to find the coordinates of a point that divides a directed line segment in a given ratio.
154.e: Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).	i. use coordinates to compute perimeters of polygons and the areas of triangles and quadrilaterals.
	ii. decompose complex figures whose vertices can be located on a coordinate plane into simple shapes to determine perimeter and/or area.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	be corrected in revision. The student may revise the selected artifact or submit a different artifact.	

Annotated Examples M154 Coordinate Geometry (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

Cities have begun to offer micro-mobility options like shared electric scooters and bikes in hopes of meeting the transportation needs of their communities. For this task, you will explore the limits and possibilities of using such devices.

Use Google Maps to locate at least three places of interest. Some options could include your school site, a place you go to often, or a place of work. Use technological tools, like Desmos or GeoGebra, or transparency paper to superimpose a coordinate system over your map.

1. *If you were to use an electric scooter that can travel 35 miles on a single charge, would you be able to make the trip from the first place of interest to the second, and return back to the first point of interest? Justify your response using words, numbers, and symbols.*
2. *How long would it normally take to make that round trip using other modes of transportation you have access to (car, bus, or walking)? How long do you estimate it would take to make the trip on a scooter?*
3. *What are the advantages and disadvantages of offering micro-mobility options?*

Sample Learning Experience

Give students an opportunity to begin the task independently, then invite them to work in pairs to share ideas and discuss strategies for completing the task. As students are working, monitor and identify solution paths to share with the whole class. Note some students may superimpose a coordinate system using tracing paper or web-based applications like Desmos or GeoGebra. Some students may center their map, as illustrated below, while others may position it in the first quadrant.

Here are some questions to facilitate the discussion of Question 1:

- *What do you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What are some commonalities and differences among the various approaches shared?*
- *How might you determine if there's enough charge to make the round trip? What information do you need to figure out in order to be confident about your answer?*
- *What can you take from this approach to apply to your own work?*

Provide additional time for pairs to revise their work and/or continue with completing the task using the ideas that surfaced.

As students start Question 3, consider offering tools for them to conduct research. Here are a few websites where they can learn more about micro-mobility:

- *Micro mobility: A Travel Mode Innovation | FHWA*
- *The Environmental Impacts of E-Scooters*
- *Micro mobility: moving cities into a sustainable future | EY*

Offer students the opportunity to work in pairs or small groups to complete the task. Give the option of using chart paper, Google Slides, or other creative outlets to showcase their work. Give students the opportunity to return to their work to revise, as necessary.

In this example, students are:

- given a cognitively demanding task. It requires students to analyze a situation, create a graphical representation of a local map, and determine the distance between identified points of interest.

This activity encourages students to reason about using graphs and determining distances to solve problems (LP 1).

- working individually and collaboratively in pairs to complete the task, allowing for social interaction among students. They are also encouraged to ask questions to better understand the thinking process (LP 2).
- building conceptual understanding through reasoning, as they decide how to situate a coordinate system onto a local map and whether a round trip is possible by determining distances. Students also analyze the benefits of using micro-mobility modes of transportation by determining how long it would take to make the round trip (LP 3).
- encouraged to have agency, as they choose the points of interest on their trip with the scooter, select approaches to their solution pathway, and make their own simplifying assumptions (LP 4).
- given space to reflect on how the use of a scooter to make a potential trip impacts themselves and others (LP 5).
- engaging with information from an online source and potentially leveraging a graphing application to complete the task (LP 7).

Example 2

Students are given a problem like this:

Part 1

According to Merriam-Webster's online dictionary, surveying *is defined as "a branch of applied mathematics that is concerned with determining the area of any portion of the earth's surface, the lengths and directions of the bounding lines, and the contour of the surface and with accurately delineating the whole on paper" (Merriam-Webster Dictionary).*

The image above is that of an ancient Egyptian map known as the Turin papyrus map. It dates back to about 1150 B.C. (A.F. Stone p. 117) and was discovered between 1814 and 1821 (Harrell).

- *Study the map and consider what things it might capture. Read more about this map at Turin Papyrus Map from Ancient Egypt. How much of what you anticipated surfaced in the reading? What surprised you?*

Part 2

Read the following excerpt from Crest of the Peacock *(Joseph, 2010):*

> *It is worth remembering that up to 1350 BC the territory of Egypt covered not only the Nile Valley but also parts of modern Israel and Syria. Control over such a wide expanse of land required an efficient and extensive administrative system. A census had to be taken, taxes collected, and large armies maintained. Agricultural requirements included not only drainage, irrigation, and flood control but also the parceling out of scarce arable land among the people and the construction of silos for storing grain and other produce. Herodotus, the Greek historian who lived in the 5th century BC, wrote that*
>
>> *Sesostris [Pharaoh Ramses II, c. 1300 BC] divided the land into lots and gave a square piece of equal size, from the produce of which he exacted an annual tax. [If] any man's holding was damaged by the encroachment of the river ... The King ... would send inspectors to measure the extent of the loss, in order that he might pay in future a fair proportion of the tax at which his property had been assessed. Perhaps this was the way in which geometry was invented and passed afterwards into Greece (Herodotus, 1984, p. 169).*
>
> *He also tells of the obliteration of the boundaries of these divisions by the overflowing Nile, regularly requiring the services of surveyors known as harpedonaptai (literally "rope stretchers").*

Rope stretchers used ropes knotted at regular intervals as a measurement tool in their line of work.

1. *For this part of the task, you will use tools of ancient rope stretchers, string, and a stick, to equally divide a rectangular piece of land into individual square lots. The land is bordered by a river along one side. Work to develop a general approach to this task.*
2. *Imagine the water level rises and floods part of the land. How would you go about redistributing the land to ensure each individual is again allotted an equal-sized square?*

Sample Learning Experience

Part 1

Invite students to read the definition of *surveying* and ask:

- *Why might someone want to determine the area, lengths, directions of bounding lines, and contours of land on paper?*
- *How would you describe what surveying is? What do you think the purpose is?*

Then share the image of the Turin Map (source Turin Papyrus Map from Ancient Egypt by Dr. James A. Harrell). Ask students to share what they notice and what they wonder about what surveyors captured on this map. Give them time to explore additional resources to confirm their predictions and to surface additional questions. Listen for students who may ask: "How did ancient Egyptians take measurements?" and amplify that question to frame the next part of the task.

Offer students additional resources to further explore cartography and land surveying in ancient Egypt:

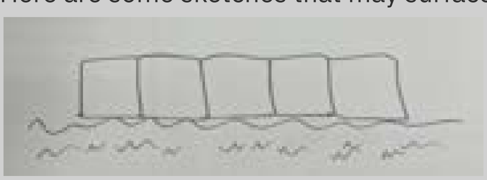
- Surveying in Ancient Egypt
- A.F. Stone: Egyptian Cartography

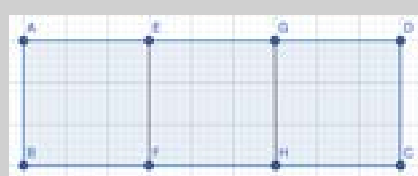
Part 2

Invite students to read the excerpt from *Crest of the Peacock*. Facilitate the reading by using a 3 Reads protocol:

- Read 1: Read the passage aloud and ask students to share what this reading is about. Record responses in a space where the class can reference.
- Read 2: Read the passage a second time and ask students what the land might look like after Sesostris' surveyors do their job. Encourage them to brainstorm with a partner as they draw a sketch. Select a couple of sketches to share with the class. Invite classmates to ask clarifying questions and revise.
- Read 3: Read the passage a third time and ask students to anticipate how someone might go about deciding how to distribute the land. Record responses in a space where the class can reference.

Here are some sketches that may surface:





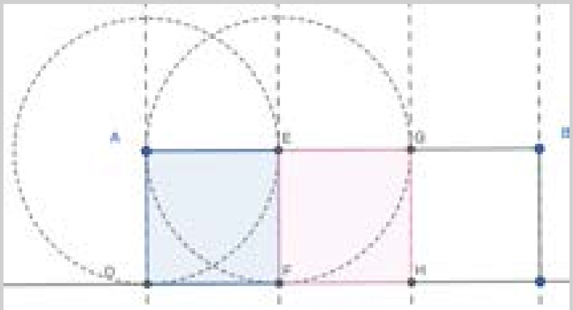
Give students time to work on Question 1. Ensure they have access to string and a straight edge of sorts (instead of a stick).

Monitor and select approaches that simulate use of a compass and straight-edge constructions. Invite students to share and learn more about each other's approaches. Ask:

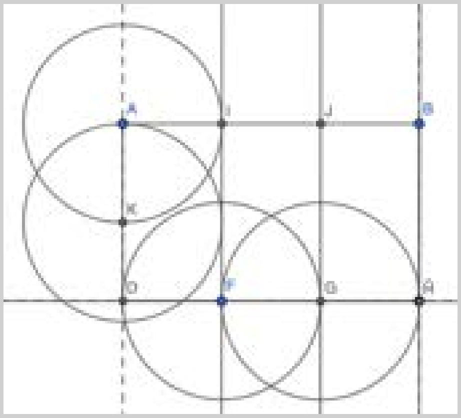
- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *Do you think the model is reasonable? Why or why not?*
- *What can you take from this approach to apply to your own work?*

Once a student surfaces an idea that brings to mind the use of a compass, encourage students to consider using tools like Geogebra.org to continue exploring approaches to this prompt.

Here are some possible approaches that may surface:



(Link to file: https://www.geogebra.org/geometry/czrz7vnd)



(Link to file: https://www.geogebra.org/geometry/sm5v88cc)

Discuss approaches to Question 1 before giving additional time to approach Question 2. Consider posing these questions for reflection between the two prompts:

- *What changes when the water level rises?*
- *What impact does that have?*
- *What about the approach you've developed do you think will still hold?*

- *What might you need to reconsider?*

Give students time to test their ideas and to develop responses before holding a whole class discussion. Monitor and select various approaches to these prompts.

Give time for students to revise and modify their responses.

In this example, students are:

- given a cognitively demanding task. It invites students to use the context of dividing a rectangular piece of land into equal-sized squares. In doing so, students will work to locate a point on a directed line segment between two points that partitions the segment in equal-sized segments (LP 1).
- set up to have social interactions, as they work in small groups and discuss their work with each other. These interactions may happen live in a physical classroom or virtually (LP 2).
- given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. Students engage in iterative cycles of exploration as they work to solve the problem (LP 3).
- able to explore skills and dispositions as they work to solve a complex problem. As they engage in this work, students have the opportunity to develop identities and interests in using mathematics to solve problems that require making deliberate simplifying choices in order to arrive at a solution (LP 4).
- able to view mathematics as a human endeavor across centuries. In exploring artifacts and the history of ancient Egyptian surveyors, students have the opportunity to see the use of simple tools for developing mathematical solutions to complex problems (LP 5).
- using technology as a tool, as they employ graphing software, string, and/or compasses to perform their analysis (LP 7).

Example 3

Students are given a problem like this.

Cesare Marchetti, a physicist and systems analyst, proclaimed that people are willing to commute from their homes for about half an hour each way. This idea is known as the Marchetti Constant.

Ancient Rome, around 275 CE, was a "walking city," meaning that the main mode of transportation was by walking. The map below illustrates an outline of the city during that period.

(Source: The Commuting Principle That Shaped Urban History - Bloomberg)

Use geometric concepts and tools to determine the degree of accuracy of the model outlined in purple in the diagram above. Aspects to explore and express in mathematical terms are as follows:

1. *What is the approximate diameter of the city? Why does that make sense, in terms of what you know about typical walking rates?*
2. *What is the approximate area of ancient Rome?*

Running water into homes was a luxury in ancient Rome. Therefore, Roman citizens were provided potable water by way of strategically placed water fountains throughout the city. It is believed that water fountains were located within a 50-meter radius from each other (Water System of Ancient Rome).

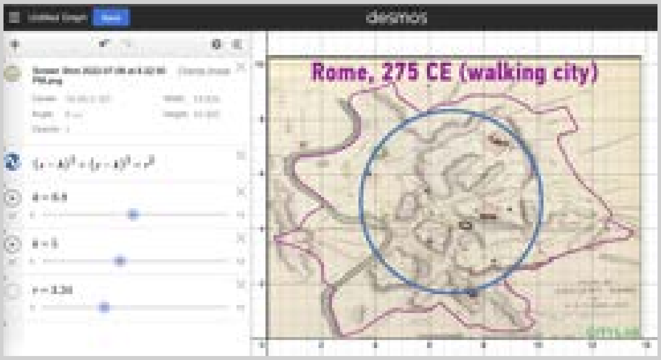
3. *Based on the work you've done so far, about how many water fountains were present in the city at that time?*

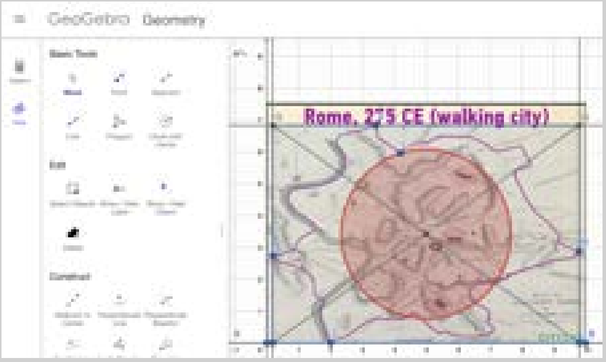
Extension

Explore the historical roots of the city you live in. How has the radius of the city changed over time? Does the Marchetti Constant hold?

Sample Learning Experience

Give students an opportunity to begin the task independently, then invite them to work in pairs to share ideas and discuss strategies for completing the task. As student pairs work, monitor and make note of solution paths to share with the whole class. Note that some students may superimpose a coordinate system using tracing paper or web-based applications like Desmos or GeoGebra.



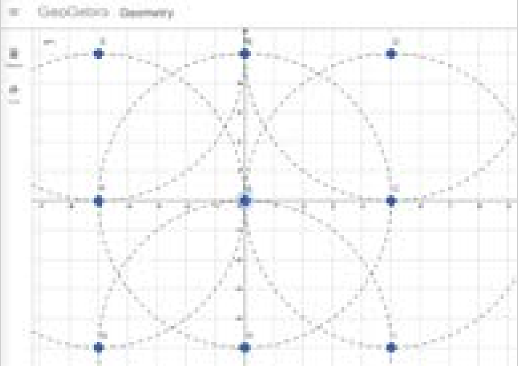


Here are some questions to facilitate the discussion of Questions 1 and 2:

- *What do you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What are some commonalities and differences among the various approaches shared?*
- *How might you determine the approximate area of the city?*
- *What information do you need in order to reach some conclusions?*
- *What can you take from this approach to apply to your own work?*

Provide additional time for pairs to revise their work and/or continue with completing the task using the ideas that surface.

As students start Question 3, monitor and select approaches they take to make sense of this. One sample approach is illustrated below. Encourage students to share and challenge approaches.



Provide additional time for pairs to revise their work and/or continue with completing the task using the ideas that surface.

In this example, students are:

- given a cognitively demanding task. It invites students to explore the idea of the Marchetti Constant and use their knowledge of coordinate geometry, circles, area, and rates to make sense of its application (LP 1).

- set up to have social interactions, as they work in small groups and discuss their work with each other. These interactions may happen live in a physical classroom or virtually (LP 2).
- given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. Students engage in iterative cycles of exploration as they work to apply their knowledge of circles and polygons, area, ratios, and coordinate geometry (LP 3).
- able to explore skills and dispositions as they work to solve a complex problem. As students engage in this work, they have the opportunity to develop identities and interests in using mathematics to make sense of real-life contexts and are required to make deliberate simplifying choices in order to arrive at a model (LP 4).
- able to view mathematics as a human endeavor across centuries. In exploring artifacts and the history of ancient Rome, students have the opportunity to see the use of simple tools for developing mathematical solutions to complex problems (LP 5).
- using technology as a tool, as they employ graphing software, string, and/or compasses to perform their analysis (LP 7).

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M155 Right Triangle Trigonometry

Badge Catalog Description
How can you estimate the height of a tree? How can you create an accurate blueprint for a new building? How can similarity shed light on relationships between angles and sides in right triangles? Trigonometry is a powerful tool that illuminates geometric relationships and helps us solve problems involving right triangles. In M155 Right Triangle Trigonometry, you will develop an understanding of what trigonometric ratios are and how they can be used to solve real-world and mathematical problems. You will use technology as a tool to calculate trigonometric ratios and find missing measurements in diagrams. Right triangle trigonometry is useful for careers in a variety of fields, like the sciences, engineering, forestry, criminology, and architecture. Suggested prerequisites for this badge: M153 Reasoning and Proof through Similarity.

The M155 Right Triangle Trigonometry badge focuses first on developing students' conceptual understanding of the sine, cosine, and tangent ratios through repeated investigations and reasoning using similarity (MP 8). Students make sense of these relationships by iteratively examining right triangles and making conjectures about the relationships between angles and side lengths. Though M155 is not a modeling badge, emphasis is also placed on using the three trigonometric ratios to solve real-world and mathematical problems. Students use technology as an aid and come to see right triangle trigonometry as a valuable problem-solving tool.

As students engage in M155 Right Triangle Trigonometry, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M155 badge.

M155 Content and Practice Expectations

155.a	Use similarity to explain the meaning of trigonometric ratios.
155.b	Explain and use the relationship between the sine and cosine of complementary angles.
155.c	Use trigonometric ratios to solve right triangles in applied problems.

Learning Principles

In M155, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M155

A key component of M155 is an emphasis on developing a deep conceptual understanding of trigonometric ratios. Typically when students encounter trigonometric ratios, instructional time is focused on memorization; students memorize each ratio and key values of ratios, which can be done without an understanding of what they mean or why they are constant. By contrast, in M155, students should see right triangle trigonometry as a topic that is logically connected to ideas they have previously encountered. In particular, they should:

- explore measurements of right triangles using rulers, protractors, and other tools (LP 7).
- use repeated reasoning to make generalizations about right right triangles (LP 3).
- be able to explain what trigonometric ratios are using similarity and use that argument to define trigonometric ratios as a property of angles, whose values remain constant (LP 3).

When engaging with trigonometric ratios, student time can also be focused on solving simple one-step problems involving trigonometric ratios, frequently without any contextual meaning. The contextual problems are also often unrealistic. Solving these problems can often involve working with cumbersome tables of ratio values or spending time calculating or solving equations by hand. In M155 by contrast, students should be exposed to multistep problems that involve real-world applications and that use technology as a tool for problem solving. In particular, students should:

- solve multistep contextual problems using trigonometric ratios and explain why solutions are reasonable (LP 1).
- develop equations that represent geometric relationships and explain why they are valid representations (LP 3).
- engage with their peers to explain what a particular trigonometric ratio means and why it is relevant to a problem-solving context (LP 2).
- engage in frequent experiences where they must make sense of given information, construct diagrams, test solution methods, and revise their thinking (LP 1).
- solve problems that explore how right triangle trigonometry is used in different careers (LP 6).
- engage in learning experiences that emphasize reasoning and explanation over performing computations with trigonometric ratios (LP 2).
- employ a variety of tools, such as calculators and geometry software, to solve problems involving trigonometric ratios (LP 7).

Evidence of Learning

In M155, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts to present evidence of their learning related to the badge content and practice expectations.

Content and Practice Expectations	Indicators Choose an artifact where you...
155.a: Use similarity to explain the meaning of trigonometric ratios.	i. measure side lengths and angles of multiple triangles and describe patterns you see.
	ii. explain why trigonometric ratios remain the same for different-sized right triangles.
155.b: Explain and use the relationship between the sine and cosine of complementary angles.	i. illustrate the relationship between the sine and cosine of complementary angles.
	ii. use the relationship between the sine and cosine of complementary angles to solve problems.
155.c: Use trigonometric ratios to solve right triangles in applied problems.	i. use trigonometric ratios to find unknown side lengths in problems.
	ii. use trigonometric ratios to find unknown angles in problems.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

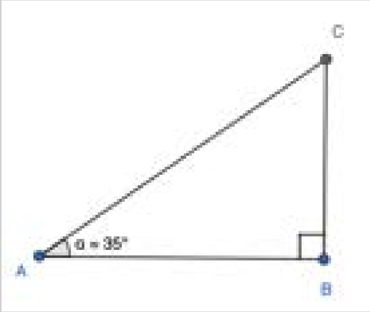
Annotated Examples M155 Right Triangle Trigonometry (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a task like this:

Right triangles have surfaced in previous work: slope, finding areas of complex shapes, and indirect measurement. In this task we will explore ratios of sides of right triangles in order to expand ways we can work with right triangles.



1. *Use measurement tools or dynamic geometry software such as GeoGebra or Desmos to construct a right triangle ABC, with a right angle at B and a 35° angle at A.*

2. *Here is some terminology used to describe the relationship between the sides of a right triangle and the reference angle, which in this case is angle A.*
    - *Side $\overline{AC}$ is the hypotenuse of this right triangle,*
    - *Side $\overline{BC}$ is the side opposite angle A, and*
    - *Side $\overline{AB}$ is the side adjacent to angle A.*

    Take a moment to study the image provided. What makes sense to you about how each side is referred to?

3. *Measure the three sides of the triangle and record the lengths in a table. Please use the terminology offered above to communicate measures.*

4. *Find each of the ratios listed below.*
    a. $\frac{\text{length of leg opposite angle } A}{\text{length of leg adjacent to angle } A}$
    b. $\frac{\text{length of leg opposite angle } A}{\text{length of the hypotenuse}}$
    c. $\frac{\text{length of leg adjacent angle } A}{\text{length of the hypotenuse}}$

5. *If you were to compare the measurements you found with that of your classmates, how do you think those measurements will compare? Are there any relationships between the side lengths you anticipate will still hold? Why does that make sense?*
6. *If you were to compare the ratios you found with that of your classmates, how do you think those ratios would compare? Why does that make sense?*
7. *Do you think the relationships you've noticed hold true for any right triangle? Construct additional right triangles to demonstrate your conjecture.*

(Adapted from Interactive Math Program Year 1 Shadows Unit)

Sample Learning Experience

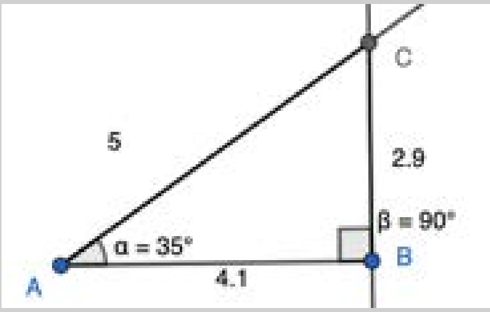
Begin by giving students quiet time to make sense of the information given and process the task they have to complete. Provide them with physical tools, such as graphing paper, transparencies or patty paper, ruler and protractor, and/or access to dynamic geometry software such as GeoGebra or Desmos.

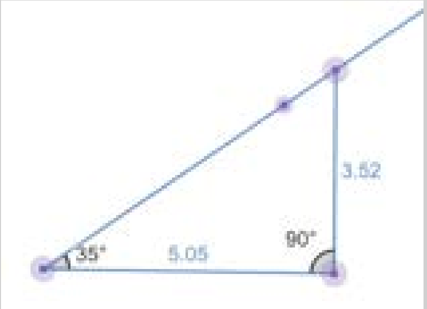
After some individual time, encourage students to discuss progress with a partner or small group. Encourage students to make their thinking visible.

Initially, focus a conversation on how students went about constructing their triangles. Consider using some of the following questions to facilitate the discussion:

- What strategies led to success in your effort to construct a right triangle with the given conditions?
- What new insights do you have about the work of constructing right triangles?
- If students used dynamic software to complete construction, select students with varied approaches to this work. Look for students who used construction tools to ensure they had 90° and 35° angles, as well as students who may have not controlled for that. Begin the conversation by asking students, *"What happens if we drag some of these points?"*
    - Invite students to attempt to drag points to recognize what elements of their triangle remain constant and which don't.
    - Follow up with additional questions to invite students to reflect on their construction approach; for example, some approaches make it so that either or both of the 90° and 35° angles remain constant.

The image on the left illustrates a student's approach using GeoGebra. The image on the right illustrates a student's approach using Desmos.





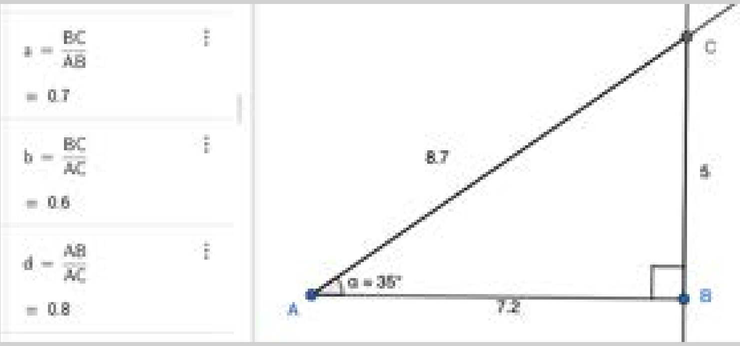
Give students time to revise their constructions as needed before proceeding to the next stage of the investigation. As students enter the next stage of the investigation, do hold space for students to have some independent work time before they collaborate with classmates.

For the second prompt, consider using the Math Language Routine 1: Stronger and Clearer Each Time (MLR 1) to give students an opportunity to learn from each other. The prompt in focus for this iteration of MLR 1 is: *When given a right triangle and an angle to focus on, how can you tell which leg of the triangle is the leg opposite of the angle? ...leg adjacent to the angle? ...hypotenuse?* A possible way to consolidate this conversation is to use Math Language Routine 2: Collect and Display or create an anchor poster using student language.

For the last prompts, consider creating a poster where students can contribute their data, similar to the table below, as classmates continue to work through the rest of the prompts.

Student	$\frac{length\ of\ leg\ opposite\ A}{length\ of\ leg\ adjacent\ A}$	$\frac{length\ of\ leg\ opposite\ A}{length\ of\ hypotenuse}$	$\frac{length\ of\ leg\ adjacent\ A}{length\ of\ hypotenuse}$

As students develop responses for Questions 5 and 6, monitor for, select, and share student work that embodies different approaches and levels of formality for discussion. Note and select various levels of formality students are comfortable with as they share their work with each other. Look for students who use arguments grounded in the idea of similarity, as well as students who utilize technology to generate multiple examples, as illustrated in the GeoGebra screenshot below.



As students share their reasoning with each other, consider the following questions:

- *What are some things you appreciate about this explanation?*
- *What questions does this explanation surface for you?*
- *What can you take from this explanation to apply to your own work?*

To engage students in a discussion of Question 7, consider selecting students who constructed additional right triangles with different reference angles, as well as those who approached this by focusing on the measure of <C being 55°.

Following the sharing and discussion, give students an opportunity to revise their work and share with each other again. Focus this second discussion on changes or adjustments that they have made based on their peers' work.

In this example, students are:

- given a cognitively demanding task. They are asked to construct right triangles that satisfy given criteria to generate the three basic trigonometric ratios: sine, cosine, and tangent. Students then have an opportunity to justify why these ratios will be approximately the same when compared to classmate's results (LP 1).
- engaging with their peers to describe the rationale behind the terminology used when describing side lengths of a right triangle, relative to the given angle (LP 2).
- building conceptual understanding as they engage in reasoning on why trigonometric ratios of a given triangle remain the same, regardless of the lengths of the sides of the triangle. This builds the foundation for understanding that trigonometric ratios, themselves, are a function of an angle (LP 3).
- encouraged to have agency, as they choose their solution pathways and are given space to reflect on how their approach might change based on engagement with their peers' work (LP 4).
- given an opportunity to see themselves as mathematicians, as their methods for generating data to support conjectures are central to this investigation (LP 5).
- given an opportunity to use geometry software to experiment with changes to the side lengths of the triangle or the given different reference angles (LP 7).

Example 2

Students are given a scenario like this:

Our phones offer tools that we can use to measure things without the use of traditional tools, like measuring tape or yard sticks. Today, you will determine the accuracy of such tools. You will need some classmates to work with, a cell phone, and access to a measuring app. This short article, 10 Best Measurement Apps for Android and iPhone, by Beebom offers some options.

Your task:

1. *Select an object to use for this task. Consider a flagpole, bell tower, or basketball basket.*
2. *Use a smartphone's measurement tool to measure the height of the object from at least three different positions.*
    a. *How do the measurements compare? If there is a difference in the heights, what do you think accounts for the differences you see?*
3. *Determine the height of that object using more traditional techniques or research. How do the measures you got from using the phone compare to the numbers you got from indirect measurement using traditional tools or from your research?*
4. *Select other objects to go through this process.*
5. *Based on your findings, how would you describe the accuracy of the smartphone's measurement app?*
6. *Conduct some additional research to determine how these phone apps work. What recommendations do you have for changes or improvements?*

Prepare to report your findings.

Sample Learning Experience

Based on what you know about your students, you may choose to launch this task by asking students, "*Who has used a measurement app on their phone?"* Following this question, invite a student or two who has used such tools to demonstrate how they work. Alternatively, you can launch this task by showing this short video clip: "Did you know your Smartphone could do THIS!?" by @Mrwhosethe boshShorts. Share with students that their task is to determine the accuracy of their smartphone measuring tool.

As students work on the first two prompts of this task, monitor and select students or student groups to share their preliminary approach to this task with the whole group. Look for groups who are deliberate about the horizontal positions they are using, as well as students who take into account their ability to frame the object (lighting, camera's ability to capture the object and key points for measurement).

Images below illustrate some of the measurements students might make with their smartphone measuring apps:

As students work, consider some of these questions to learn more about their approach:

- *How did you decide where to stand? How did you decide what other positions to try?*
- *What have you noticed about how this app works? What impact do you think that has on what and how you measure things?*

As students share their approach with classmates, invite them to make sense of each other's work. Consider the following questions:

- *What are some things you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again. Focus this second discussion on changes or adjustments they have made based on their peers' work.

As students return to work on the remaining prompts, monitor and select students who use similar triangles and/or trigonometry as forms of indirect measurement, as well as other mathematically promising approaches. Ensure tools for measuring angle of elevation are readily available for use or to be assembled.

As they share their approach with classmates, invite students to make sense of each other's work. Consider the following questions:

- *What are some things you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again.

Encourage students to consider sharing their findings and suggestions for improvements with app developers. Invite students who hear back from developers to share the impact of their work with the class.

In this example, students are:

- given a cognitively demanding task. They are asked to execute a plan for determining the accuracy of smartphone measurement apps. Students also have the opportunity to use indirect measurement techniques to compare measurement results (LP 1).
- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions of each other to better understand the thinking process, as well as given space to learn from their peers (LP 2).
- building conceptual understanding of the use of trigonometric ratios through indirect and actual measurement (LP3).
- given space to develop agency, as they choose their solution pathways and reflect on how their approach might change based on engagement with their peers' work (LP 4).
- given an opportunity to see themselves as mathematicians, as their methods for generating data to support conjectures are central to this investigation (LP 5).
- using technology as a tool and learning to determine accuracy of such tools (LP 7).

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M201 Function Concepts

Badge Catalog Description

Are you ready to deepen your knowledge of functions? These are versatile modeling tools that allow us to represent any situation in which a change in one measurable quantity results in a change in a related measurable quantity. Understanding functions allows us to create, analyze, and interpret them in a variety of contexts.

In M201 Function Concepts, you will extend and deepen your mathematical understanding of linear, exponential, and quadratic functions. You will understand sequences as functions and explain relationships between explicit and recursive representations. Additionally, you will reason and generalize ideas about transformations of functions, using technology as an aid to analyze your graphs while also exploring ideas of domain and range and constructing inverse functions. Understanding function concepts sets the stage for further work with more complicated functions, including polynomial, rational, exponential, and logarithmic functions. Function concepts are useful for careers in a variety of fields, like the sciences and engineering.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

The M201 Functions Concept badge puts mathematical reasoning at the center of how students engage with the various concepts associated with functions. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with functions. According to the Grade 8, High School, Functions progression, “students should develop ways of thinking that are general and allow them to approach any type of function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary. For example, they should see linear and exponential functions as arising out of structurally similar growth principles; they should see quadratic, polynomial, and rational functions as belonging to the same system” (Common Core State Standards Writing Team, 2013, p.7).

According to *Catalyzing Change in High School Mathematics*, “The ability to recognize and reason about particular characteristics of a class of functions is essential to understanding and using functions to model different phenomena appropriately and to solve problems” (National Council of the Teachers of Mathematics, 2017, p. 53). By building on functions, students gain insights into the mathematical relationships between quantities in play.

As students engage in M201 Function Concepts, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M201 badge.

M201 Content and Practice Expectations

201.a	Understand the concept of a function and use function notation.
201.b	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
201.c	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
201.d	Build new functions from existing functions.
201.e	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).
201.f.	Build a function that models a relationship between two quantities.

Learning Principles

In M201 Function Concepts, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M201

Whereas a typical instructional unit on functions might begin with a focus on students visualizing relations and functions, in M201, students should:

- collaboratively explore the graphs of functions (linear, quadratic, exponential, logarithmic, square root, cube root, polynomial, and piecewise-defined functions, including step functions and absolute value functions) with opportunities to discuss and gain deep understanding of the key features of each of the graphs (LP 2).
- communicate their understanding of functions and the relationship between domain and range (LP 3).
- examine the structure of equivalent forms of an expression representing a function to gain insights about the different properties of functions (LP 3).

- regularly encounter real-world tasks involving functions (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical (LP 3).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).
- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, algebraic, numerical, graphical. As examples, students should engage with the following:
    - Analyze and explain the relationship between parameters in a function and the features of its graph to graph a function (LP 3).
    - Reason about and recognize that sequences are functions (LP3).
    - Reason about and explain the real-world meaning of key features of a graph.
    - Make connections between the different representations of a situation.
    - Compare properties of two functions each given in a different representation (LP 3).
- use open-ended tasks to interpret data, analyze graphs, and reason about the context (LP 3).
- have opportunities to use reasoning to relate the algebraic form of a function to its graph and key features (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that functions can authentically involve real-world situations (LP 6).

Often, coursework with functions is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use spreadsheets and graphing software to allow for focus on understanding the effect on the graph when replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, and $f(x + k)$ for specific values of k, seeing patterns, forming generalizations, and testing conclusions (LP 7).

Evidence of Learning

In M201 Function Concepts, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence,
AND
(2) Concepts and Skills Assessment.

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
201.a: Understand the concept of a function and use function notation.	i. demonstrate understanding of the following language of functions: If *f* is a function and *x* is an element of its domain, then *f(x)* denotes the output of f corresponding to the input x. The graph of *f* is the graph of the equation *y* = *f(x)*.
	ii. recognize that sequences are functions, sometimes defined recursively, and their domain is a subset of the integers.
201.b: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.	i. graph linear and quadratic functions and show intercepts, maxima, and minima.
	ii. graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
	iii. graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
	iv. graph exponential functions showing intercepts and end behavior.
201.c: Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.	i. rewrite expressions, using processes such as factoring and completing the square in a quadratic function, to show zeros, extreme values, and symmetry of graphs.
	ii. use the properties of exponents to interpret expressions for exponential functions.
201.d: Build new functions from existing functions.	i. identify the effect on the graph of replacing *f(x)* by *f(x) + k*, *k f(x)*, *f(kx)*, and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
201.e: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal	i. compare properties of two functions that highlight two quantities or features of interest.

Content and Practice Expectations	Indicators Choose an artifact where you...
descriptions).	
201.f: Build a function that models a relationship between two quantities.	i. identify and/or define the variables of interest in a given situation or data set that demonstrate a functional relationship.
	ii. build a function that models the relationship between two quantities in a situation.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M201 Function Concepts (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Student are presented with a scenario like this:

Part A: Speeding in Massachusetts

A speeding ticket fine in Massachusetts is $100 for any speed up to 10 miles above the posted speed limit. The fine increases by $10 for each additional mile an hour after the first 10.

[Source: Speeding Ticket FAQs | Massachusetts Speeding Ticket Lawyer]

1. *Create a graph of this situation where the number of miles over the posted speed limit is on the horizontal axis and the cost of the ticket is on the vertical axis.*
    a. *Does this graph represent a function?*
    b. *What domain and range are reasonable for this function? How do you see these in the graph?*
2. *Suppose the posted speed limit is 65 miles per hour. Create a "formula" that takes the driver's speed at the time of incident in that zone and generates the cost of the driver's speeding ticket.*
    a. *Explain how the formula you created works. What is the input? What is the output?*
    b. *Define any variables you use and illustrate how the formula(s) work.*

Part B: Speeding in New York

Fines for speeding vary from state to state. In New York, speeding penalties are determined as follows:

Penalties for Speeding

Speed	Minimum fine	Maximum fine	Possible prison time
up to 10 mph over	$45	$150	not more than 15 days
more than 10 mph over - less than 30 mph over	$90	$300	not more than 30 days
more than 30 mph over	$180	$600	not more than 30 days
inappropriate speed	$45	$150	not more than 15 days

Source: Penalties for Speeding | Governor's Traffic Safety Committee

1. *Create a graph of this situation that illustrates the relationship between quantities.*
    a. *What is interesting about your graph?*
    b. *What do some of the interesting features of the graph mean relative to the context?*

2. *How does New York's Speeding Penalty system compare to Massachusetts' system?*
    a. *What are your thoughts about this?*

Part C: Extending the learning:

Learn about your state's or another state's or another state's system for determining speeding penalties.

  b. *Is it possible to represent the system graphically? Using a formula of some sort? Why or why not?*
  c. *How does your state's system compare to the systems in New York and Massachusetts?*
  d. *What are your thoughts on this?*

Sample Learning Experience

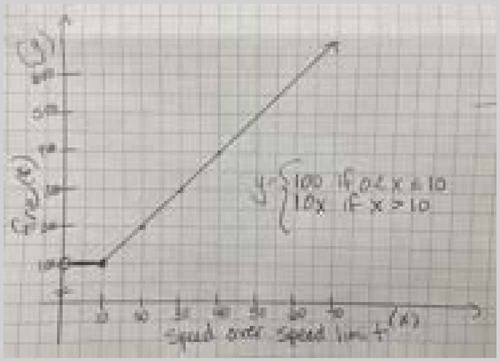
Launch this task by engaging students in a conversation to make sense of driving laws, speed limits, and penalties for going over the speed limit. Some questions you can pose are as follows:

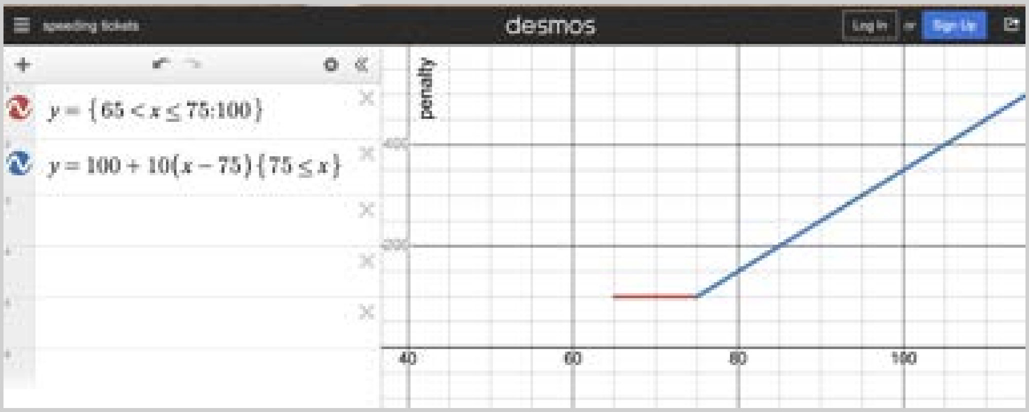
- *How does a driver determine how fast to go at any point in time?*
- *How does a driver know the maximum speed for any road they're on?*
- *What are some consequences for going over the speed limit?*

Introduce the prompt and provide time for students to consider the questions posed. Give students time to work independently and develop a response. Ensure students have access to multiple tools they can use to develop a response.

Invite students to partner with classmates and share their responses. Provide time for students to revise their justifications using the information gained during these discussions.

Some possible approaches are as follows:





Monitor and select various solution methods to share with the class as students work with classmates. Have selected students present their responses—consider using a document camera for students to show actual work—and explain their justification. As discussion takes place, invite classmates to make sense of and build on each other's responses, allowing for students to clarify and defend their response as the conversation ensues. Ask:

- *What about this reasoning makes sense?*
- *In what ways is using a function helpful to communicate speeding penalties?*
- *What questions surface for you?*
- *Is there anything you have as an alternate explanation or point of view?*
- *What can you take from this approach to apply to your own work?*

Following the discussion, provide students an opportunity to reflect on and revise their work. Additionally, offer students tools to explore any of the questions that surface from examining the speeding penalty practices of these states. Follow a similar structure for Parts B and C.

In this example, students are:

- given a cognitively demanding task. Students grapple with determining which quantities work together to determine a function grounded in a context. Students have multiple entry points to reason about this prompt in various ways (LP 1).
- using the varied structures that give opportunities to determine an approach by working independently, then sharing with a partner or class, then revising their response based on those interactions (LP 2).

- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- given the opportunity to engage in mathematics that authentically involves a real-world situation (LP 6).
- considering how a context can be represented using mathematical tools, as well as see how the idea of a function relates to the idea of a just system. In doing so, students have an opportunity to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- using technology to represent this context and show and explain connections between an equation and a graph (LP 7).

Example 2

Students are given a problem like this:

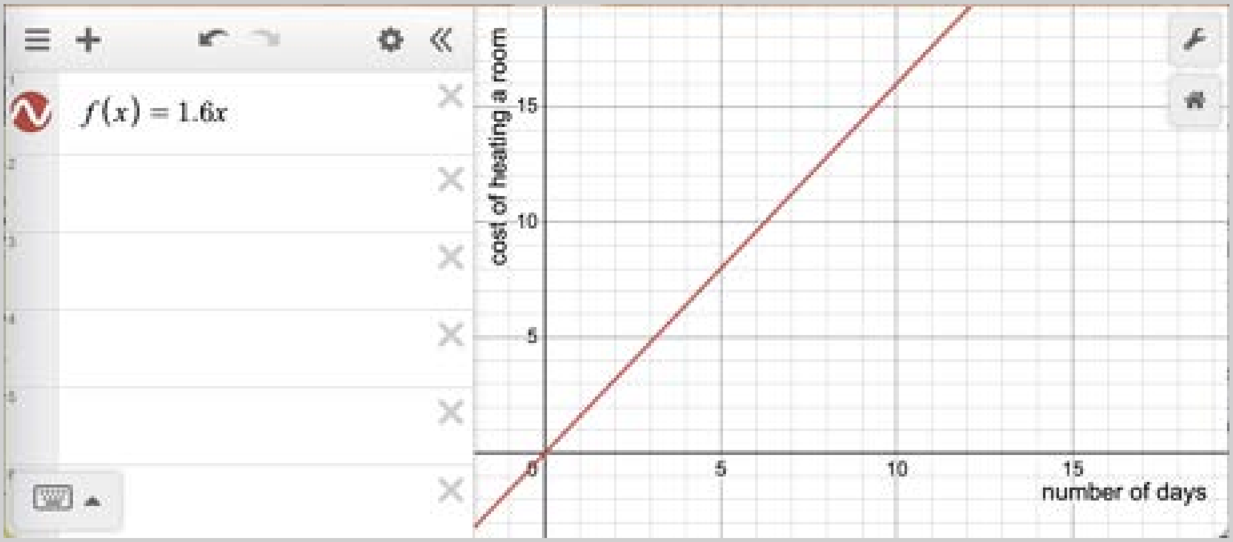
During the winter months, some schools shut down due to extreme cold conditions and the schools' inability to adequately heat their buildings. According to the Capital Gazette and Bloomberg, the Baltimore City Public School System experienced this over the course of two winters in 2017 and 2018. In an effort to help, one college student started a GoFundMe campaign to purchase space heaters, raising about $84,000. One high school student reported near freezing temperatures of $50^{\circ}F$ at the start of the day. With the use of a space heater, the classroom temperatures increased to $65^{\circ}F$. In Maryland, the estimated cost to run a heater is about $0.20 per hour.

a. *Examine the image below or explore the actual graph at https://www.desmos.com/calculator/nts1vbqygf*

   i. *How would you describe the relationship between the input and the output?*

   ii. *What does this relationship tell you about the costs of heating a room? How do the variables, slope of the line, and the equation of the function capture that relationship?*

   iii. *If a school has set up 40 classrooms each with a space heater, how would the function between the number of days and the cost of heating change? How does this impact the graph of the function? How does it impact the equation?*



b. *Consider the following graph of $y = g(x)$.*

i. *How would you describe the relationship between the input and the output?*

ii. *What values of the input are part of the domain of this function? Why does that make sense?*

iii. *What does the relationship represented below tell us about how a space heater works to warm a room?*

iv. *Given the conditions, the school leadership team decides to have administrators who are already in the building two hours before the start time preheat all the classrooms and keep the heaters running until the end of the school day.*

1. *Sketch a graph of the new function h, which gives the temperature in the school as a function of time.*
2. *Write an equation for h in terms of g. Explain why your equation makes sense.*

3. *How will the school leadership team's decision to start heating rooms earlier impact their cost for heating the school?*

Sample Learning Experience

Have students examine the following articles to set context to the situation. Lead a class discussion based on thoughts, reflections, surprises, etc. from the articles.

- Ericson Jr., E. (2018, January 7). How Baltimore Students Got Left in the Cold. Bloomberg. How Baltimore's Public Schools Got Left In the Cold - Bloomberg
- McNamara, J. (2017, January 10). Cold classrooms a hot issue in Bowie. Retrieved from Capital Gazette: Cold classrooms a hot issue in Bowie – Capital Gazette

Lead into the activity with some individual work time, where students develop an approach to the prompt, encouraging them to share and compare approaches. Monitor and select a couple of approaches to highlight and reflect on progress made at this point.

Questions that can be posed to student pairs as they work:

- *What is this context about? What connections can you make to your own experience?*
- *What are the quantities, things that can be counted or measured, involved in this context?*
- *What do you anticipate is the relationship between these quantities? Why does that make sense?*
- *What connections can you make between these ideas you shared and the first graph?*
- *What connections can you make between these ideas you shared and the first equation?*
- Press for details by asking *can you say more about...?*

Give selected pairs of students time to share responses and invite classmates to reflect on the following questions:

- *What makes sense about the ideas shared?*
- *What questions surface for you?*
- *What can you take from this discussion and apply to your own work?*

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure and relationship between the equations to determine which sets of equations have the same answer. Give students additional time to revise their work.

Additionally, offer students tools to explore any of the questions that surface from examining the case of Baltimore Public Schools.

In this example, students are:

- given a cognitively demanding task. They are asked to analyze the parameters of a scenario in order to apply it to a linear function. Students have multiple entry points to reason about the functions in play by way of multiple representations and work to describe implications of changes in parameters and considerations on the function (LP 1).

- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking processes and revise their own thinking (LP 2).
- making sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical. Additionally, students work to communicate correspondences between the various representations, deepening their understanding of the concept of a function (LP 3).
- encouraged to develop skills of expert learners, thereby developing agency, as they reflect on their approach to the prompt and incorporate new ways of thinking about this from their peers (LP 4).
- considering how a context can be represented using mathematical tools, as well as how changes in the context impact these representations. In doing so, students have an opportunity to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- given the opportunity to engage in mathematics that authentically involves a real-world situation (LP 6).

Example 3

Students are given a situation like this:

Congratulations! You have just won the lottery! You are given a choice of what your pay out will be. Would you rather receive a lump sum of $1,000,000 or annual payments spread out over some years?

a. *Without doing any calculations, which is your initial choice and why?*

Let's learn more about how lotteries work and the options offered. Take a moment to read Jaime Johnson's article "How lotteries work and 3 key steps to take if you win one" published November 1, 2022 on Business Insider. [How lotteries work and 3 key steps to take if you win one]

b. *If you choose the lump sum option and place the money in a savings account, we can use a function to represent the amount of money in the account after t years. One such function is* $f(t) = 538200 \cdot 1.0018^{t}$.

  i. *Based on what you learned from the article, explain what each of the parameters (variables and constants) in the function represent.*

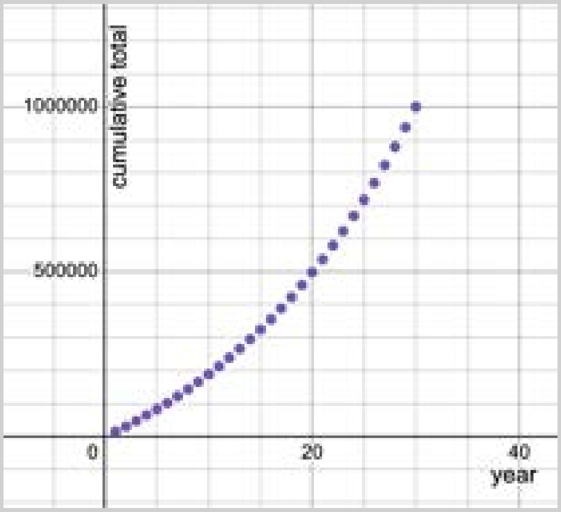
  ii. *What type of function is this? Why does it make sense to have this type of function in this case?*

c. *The option of having annual payments over time is referred to as an annuity. If you choose a lottery annuity, you agree to receive an initial payment followed by yearly payments. In the case of the Mega Millions lottery, winners agree to receive 29 yearly payments, where each payment grows by 5% from the previous year. The image below shows the cumulative total a winner of a $1,000,000 prize would have over the course of that period. You can also access the graph at this link: https://www.desmos.com/calculator/f6zy9mijdh.*

  i. *How would you describe the function that fits the annuity plan?*

  ii. *Use the link provided or a Lottery Annuity Calculator, like Lottery Annuity Calculator, to explore and identify a function that describes how the annuity program works.*

    1. *What do each of the components of your function mean in terms of the context?*



d. *Which lottery choice is more lucrative? How do the insights gained from looking at the various mathematical representations help support your response?*

Sample Learning Experience

Give students an opportunity to independently begin exploration, then invite them to work in pairs to engage in the exploration. As students work, monitor for the different ways they communicate their reasoning.

Select pairs of students to share responses. As students share their ideas, consider use of the following questions to facilitate the discussion:

- *What are you attending to as you consider how each option works? How do the various mathematical representations make that visible?*
- *If we compare the two options over shorter time frames, like two or three years, or 12 and 20 years, what do you notice? Is that enough information to make a decision about which option is more lucrative?*
- *How do each of the options grow?*

Invite classmates to reflect using the following questions:

- *What makes sense about the ideas shared?*
- *What questions surface for you?*
- *What can you take from this discussion and apply to your own work?*

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure of the various representations, the relationship between the variables involved, and additional considerations they took into account in order to determine which option is the most lucrative. Give students additional time to revise their work.

In this example, students are:

- given a cognitively demanding task. They are asked to explore and explain how the various payout options are offered to lottery winners. Within this context, students have an opportunity to make sense of exponential functions and geometric sequences and series (LP 1).

- working individually and collaboratively as they make progress through this exploration. Students are encouraged to ask questions to their peers to understand one another's thinking processes (LP 2).
- given a task that centers reasoning with multiple opportunities to make sense of payout options as well as make connections between the underlying mathematical principles in play and their corresponding mathematical representations (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- given the opportunity to engage in mathematics that authentically involves a real-world situation (LP 6).
- considering how a context can be represented using mathematical tools, as well as see how the functions can be used to make informed financial decisions. In doing so, students have an opportunity to broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5).
- using technology to represent this context and show and explain connections between an equation and a graph (LP 7).

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M202 Rational Exponents and Complex Numbers

Badge Catalog Description

How do weather forecasters create models to determine if an umbrella is needed that day? Rational exponents play an important mathematical role by connecting powers and roots, as well as in real-world situations like depreciation and inflation. Complex numbers ensure that every algebraic equation has a solution, and they make the work of designing Wi-Fi equipment, electric cars, and wind turbines easier. In M202 Rational Exponents and Complex Numbers, you will complete your study of the real number system and extend your study of numbers beyond it to the complex number system.

As you make sense of rational exponents and complex numbers, you will evaluate and simplify numerical and algebraic expressions involving radicals and rational exponents, solve simple rational and radical equations in one variable, and add, subtract, and multiply complex numbers. Additionally, you will solve quadratic equations with real coefficients containing non-real solutions and that represent complex numbers on the complex plane in rectangular form. Complex numbers are beneficial in describing many complex situations, such as the laws of electricity and electromagnetism. Both rational exponents and radical expressions are useful for careers in a variety of fields, like architecture, carpentry, electrical engineering, finance, and masonry.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

The M202 Rational Exponents and Complex Numbers badge puts rational exponents at the center of how students engage with radicals along with understanding that real numbers are a subsystem of complex numbers. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with rational exponents and complex numbers.

According to *Catalyzing Change in High School Mathematics*, “Number remains important at the high school level. In high school, students study the irrational numbers, together with the rational numbers, integrated in the various content domains. ... Together, irrational numbers and rational numbers complete the real number system, representing all points on the number line” (National Council of the Teachers of Mathematics, 2017, p. 44). Also, “The introduction of rational exponents and systematic practice with the properties of exponents in high school, widen the field of operations for manipulating expressions” (Common Core Standards Writing Team, 2013, p. 5). By building on the properties of exponents and numbers, students gain insights into the mathematical relationships between radicals and rational exponents and complex numbers with the real and imaginary numbers.

As students engage in M202 Rational Exponents and Complex Numbers, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M202 badge.

M202 Content and Practice Expectations

202.a	Reason about and extend the properties of exponents to rational exponents.
202.b	Reason about and perform operations with complex numbers.
202.c	Analyze and use complex numbers in polynomial equations.

Learning Principles

In M202 Rational Exponents and Complex Numbers, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M202

A key component of M202 is an emphasis on relationships. Whereas a typical instructional unit on complex numbers might focus on performing computations or manipulating expressions by hand disconnected from real-world contexts, in M202, students should:

- have opportunities to understand and explain why writing radical expressions in the form of rational exponents allows for easy simplification of the expressions (LP 3).
- understand and apply the properties of exponents to simplify expressions or solve problems involving radicals and rational exponents (LP 3).
- recognize that working with complex numbers is like working with polynomial expressions (LP 3).
- understand that complex numbers are useful for representing two-dimensional quantities (LP 3).
- regularly encounter real-world tasks involving polynomial equations (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical (LP 3).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).
- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, algebraic, numerical, graphical. As examples, students should engage with the following:

  - Make connections between the representation of a radical expression and an expression having rational exponents (LP 3).
  - Perform operations with radical expressions (LP 3).
  - Reason about and differentiate between a real number and an imaginary number.
  - Reason about complex numbers and perform addition, subtraction, and multiplication with complex numbers (LP 3).
  - Engage with a variety of quadratic equations to reason about having complex solutions (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that rational exponents and complex numbers can authentically involve real-world situations (LP 6). Rational exponents are useful in fields like architecture, carpentry, and masonry. Complex numbers are very useful in many interesting applications such as in physics when dealing with electromagnetism or quantum mechanics.

Often, coursework with complex numbers is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use spreadsheets to allow for focus on understanding, rather than computation or symbolic manipulation. Students should write spreadsheet formulas that are useful in such applications as compound interest or annual rate of inflation (LP 7).

Evidence of Learning

In M202 Rational Exponents and Complex Numbers, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
202.a: Reason about and extend the properties of exponents to rational exponents.	i. explain how the meaning of rational exponents follows from extending the properties of integer exponents to those values, and how this definition leads to notation for radicals in terms of rational exponents.
	ii. rewrite expressions involving radicals and rational exponents using the properties of exponents.
202.b: Reason about and perform operations with complex numbers.	i. demonstrate understanding that there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
	ii. use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
202.c: Analyze and use complex numbers in polynomial equations.	i. solve quadratic equations with real coefficients that have complex solutions.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

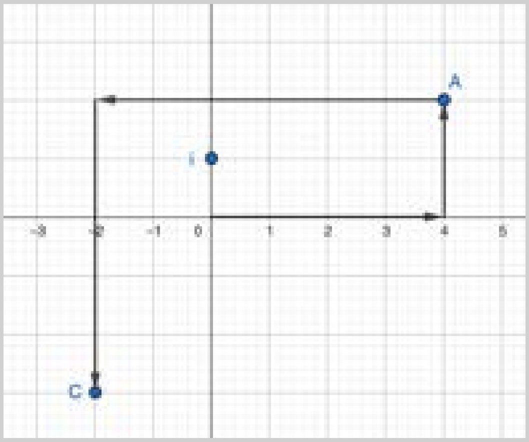
Annotated Examples M202 Rational Exponents and Complex Numbers (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are presented with a problem like this:

1. *The diagram represents the sum of two complex numbers:* $(4 + 2i) + (-6 - 5i)$.



a. Which part of the expression does Point A represent? Explain how you know.

b. What complex number does Point C represent? Explain how you know.

c. Based on the analysis above, describe how to find the sum of two complex numbers.

2. *Write and record a tutorial for a new classmate that explains how to simplify expressions that involve addition or subtraction of two complex numbers. Be sure to include a diagram that illustrates why your method works.*

Sample Learning Experience

Provide time for students to consider Question 1 and independently develop a response. Engage students using the math language routine Stronger and Clearer Each Time, as described below:

1. Have students write their response to 1c, including their justification.
2. If you have emerging multilingual students in your class, give students a minute to visualize themselves verbalizing their thoughts to a classmate.
3. Ask students to meet with a classmate and take turns to share what they wrote (about two minutes each). As one person shares, the other should pose clarifying questions.
4. Give students about one minute to reflect on this partner sharing. Encourage students to add new thinking to their original response based on what they heard.
5. Have students switch partners and repeat the process of sharing their thinking and refining their response. Repeat this process as much as time permits.

Provide time for students to revise their written justifications using the information gained during their partner-sharing time. While observing student discussions, identify varying solution methods to be presented during the whole group share out. Look for methods that invite connections between the graphical representation and a purely arithmetic approach. Have selected students present their responses—consider using a document camera for students to show actual work—and explain their justification. Focus the discussion on Question 1c. As discussion takes place, invite classmates to join the discussion. Ask:

- *How do you differentiate the real and imaginary axis?*
- *Why does each axis have a positive and negative side?*
- *Do you agree with ____ explanation of Point A? What about this explanation is clear to you? How can we strengthen the explanation?*
- *Is there anything you disagree with in this explanation??*

Provide time for students to revise their work.

Use student responses to Question 1c to launch work on Question 2. Pose a question like, "Will the method you described (1c) work for any pair of complex numbers?" Encourage students to explore concrete examples as they prepare their response to Question 2.

Consider inviting select students to share their approach at strategic points in time as they work on Question 2. As students present their ideas, invite classmates to reflect using the following questions:

- *What makes sense about the ideas shared?*
- *What questions surface for you?*
- *What can you take from this discussion and apply to your own work?*

Provide time for students to revise their work. As students continue their work, select additional pairs to share how they structured their investigation to determine a response to the question, "Does this method work for any two complex numbers?" Give students additional time to revise their work.

In this example, students are:

- engaged in a cognitively demanding task where students use their understanding of the Cartesian plane to make sense of the complex plane. Students then leverage the structure of the plane to build an explanation for the arithmetic of complex numbers (LP 1).

- given the opportunity to learn and build from each other's explanations for the arithmetic of complex numbers (LP 2).
- given the opportunity to build conceptual understanding of the underlying structure of the complex number system as they reason and work to explain how to add and subtract complex numbers. In building their mathematical argument, students have the opportunity to explore multiple cases of $(a \pm bi) + (c + di)$ (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4 and LP 5).
- encouraged to use technology to create representations of the cases they examine and the argument they work to build (LP 7).

Example 2

Students are presented with a problem like this:

Part 1

Let's think about what $a^{\frac{1}{2}}$ *means.*

1. *Select some value for* a *and develop a table to show that for some values for* a^x*,* x *is any set of integer values of your choice.*
2. *Based on the values from your table, make an educated guess about the value of* $a^{\frac{1}{2}}$*. Explain your reasoning.*
3. *Check the actual solution for* $a^{\frac{1}{2}}$*. Explain why this answer makes sense. If the answer is different from your guess above, write down two-three ideas or questions you can pose to help examine why* $a^{\frac{1}{2}}$ *has the designated value.*
4. *Based on conclusions you've reached, what do you anticipate something like* $a^{\frac{3}{2}}$ *means?*
    a. *How can you use what you see in your table or what you know about properties of exponents to explain what this means?*

Part 2

5. *Based on the discussion of Part 1, what do you anticipate the values of* $a^{\frac{1}{3}}$ *and* $a^{\frac{1}{4}}$ *will be?*
    a. *How can you use what you see in your table or what you know about properties of exponents to explain what this means?*
6. *Create an video or animated short explaining what an expression like* $d^{\frac{m}{p}}$ *means.*

Sample Learning Experience

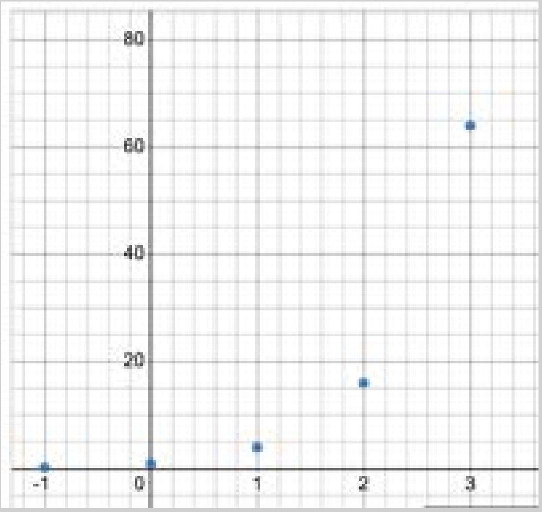
Part 1

After some individual work time, where students develop an approach to the prompt, encourage them to share and compare approaches. Students should have access to tech tools that facilitate problem solving, like spreadsheets and scientific and graphing calculators. As students share with partners, monitor and select a couple of approaches to highlight and reflect on progress made at this point. Look for student work that offers space to discuss choices students made about values to use as the base and exponents, as well as a variety of ways that students reasoned about this. Some sample approaches are illustrated below.

4^1	$4^{\frac{1}{2}}$	4^2	4^3
4		16	64

2^1	$2^{\frac{1}{2}}$	2^2	2^3
2		4	8

25^1	$25^{\frac{1}{2}}$	25^2	25^3
25		625	15625



Questions that can be posed to student pairs as they work and/or as students share with classmates:

- *How did you decide on a value of a to use?*
- *How are you selecting powers of a to use? Which powers of a are helpful in reaching a conclusion?*
- *Did you notice anything interesting about the values? What is happening to your numbers?*
- *How did you reach a conclusion about the meaning of a fraction as an exponent?*

Give selected pairs of students time to share responses and invite classmates to reflect on the following questions:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide additional time for students to revise their work, as well as to make progress on the additional prompts in Part 1. While they work, select additional pairs to share how they used the structure of the expressions and their understanding of the values, either by way of reasoning through the values in the table and/or graphs, to reach conclusions. Give students additional time to revise their work. Consider the benefits of holding whole class discussions after Question 2 and again after Question 4. Chart key findings that students can build from to transition to Part 2.

Part 2
After some individual work time, where students develop an approach to the prompt, encourage them to share and compare approaches. Consider using similar prompts as listed above to give space for students to develop a strategic approach. After a whole group discussion of Question 5, provide

additional time for students to revise their work and to develop their response to Question 6. Showcase student videos and animations. Give students an opportunity to reflect on this experience and revise their work as they see fit. Some questions for reflection are as follows:

- *What is something you learned about yourself in conducting this mathematical investigation?*
- *What is something you're proud of?*
- *What is a new insight you have about how mathematics works?*

In this example, students are:

- engaged in a cognitively demanding task. They are asked to develop a mathematical investigation where they have an opportunity to analyze the pattern of whole number exponents to make sense of fractional exponents. Students have multiple entry points to reason about these expressions in various ways (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask their peers questions to understand each other's thinking process and revise their own thinking (LP 2).
- attending to a meaning of exponents as being repeated multiplication and working to make sense of the relationships between whole and fractional exponents. In so doing, students learn to utilize various math tools to reason about and define rational exponents (LP 3).
- given space to develop skills of expert learners, thereby developing agency, as they reflect on their approach to the prompt and incorporate new ways of thinking about this from their peers (LP 4).
- given the opportunity to develop their mathematical identity as they are in the driver's seat throughout this investigation (LP 5).
- using technology as a tool for testing conjectures they develop along the way (LP 7).

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M203 Polynomial and Rational Expressions, Functions, and Equations

Badge Catalog Description

How do aerospace engineers determine the acceleration of a rocket? How do astronomers calculate the distance between a new star and the Earth? If you add, subtract, multiply, or divide two numbers, the answer will be another number. If you add, subtract, multiply, or divide two algebraic expressions, the result will be another algebraic expression: a polynomial or rational expression, to be precise. In this way, advanced algebra generalizes the familiar process of arithmetic. In the same way that arithmetic is useful for solving simple world problems, polynomial and rational functions are useful for modeling a variety of situations in contexts such as meteorology, economics, and financial planning.

In M203 Polynomial and Rational Expressions, Functions, and Equations, you will reason about and explore the interplay between algebraic expressions and the functions they define. As you learn these mathematics, you will make sense of the properties used to manipulate, add, subtract, multiply, and factor polynomial expressions. Understanding how factors and zeros of polynomials are related will aid in graphing polynomial and rational functions and solving polynomial equations by factoring and using given factorizations. Using technology tools such as computer algebra systems (CAS) will be beneficial for making sense of the features of the graph and how the graph relates to the algebraic form. Polynomials and rational expressions, functions, and equations are useful for careers in a variety of fields, like aerospace engineering, mechanical engineering, and astronomy.

Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type.

This badge is suggested as a prerequisite for: Precalculus and Calculus.

The M203 Polynomial and Rational Expressions, Functions, and Equations badge puts algebraic reasoning at the center of how students engage with polynomial and rational expressions, functions, and equations. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with recognizing and using structure in expressions, functions, and equations. “Polynomials and rational expressions come to form a system in which they can be added, subtracted, multiplied, and divided. Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers” (*High School Algebra*, Common Core Standards Writing Team, 2013, p. 7).

According to *Catalyzing Change in High School Mathematics*, “Careful consideration needs to be given to when and how technology can be used to shift the focus from learning many individual procedures for algebraic manipulations to considering multiple equivalent forms of expressions and equations, interpreting the results of manipulation, and making strategic choices about which forms of an expression or equation to use" (National Council of the Teachers of Mathematics, 2017, p. 47). Also, "Different symbolic forms of the same equation can represent different contextual interpretations and connect with different features of the graph that describe the relationship between the variables" (National Council of the Teachers of Mathematics, 2017, p. 51). By building on the multiple representations of functions, students gain insights into the mathematical relationships between quantities in play.

As students engage in M203 Polynomial and Rational Expressions, Functions, and Equations the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M203 badge.

M203 Content and Practice Expectations

203.a	Interpret the structure of polynomial and rational expressions.
203.b	Reason about and perform operations on polynomials.
203.c	Understand the relationship between the zeros and factors of polynomials.
203.d	Use polynomial identities to solve problems.
203.e	Rewrite rational expressions.
203.f	Interpret polynomial and rational functions that arise in applications in terms of the context.

Learning Principles

In M203 Polynomial and Rational Expressions, Functions, and Equations, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M203

Whereas a typical instructional unit on polynomial and rational exponents, functions, and equations might begin with a focus on students performing computations or manipulating expressions, functions, and equations disconnected from real-world contexts, in M203, students should:

- recognize that polynomials form a system analogous to the integers (LP 1).
- communicate the importance and benefit gained from writing polynomial and rational expressions in a variety of forms (LP 3).
- Solve contextual problems using equivalent forms of expressions (LP 1).

- discover and explain the connections that exist between the Remainder Theorem and their evolving understanding of what a factor is (LP 3).
- regularly encounter real-world tasks involving polynomial equations (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical (LP 3).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).
- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, algebraic, numerical, graphical. As examples, students should engage with the following:
    - Reason about and interpret expressions that represent a quantity in terms of its context (LP 3).
    - Construct rough graphs for polynomial functions by using their zeros and make sense of what the zeros mean for the graph (LP 3).
    - Use structure of polynomial and rational expressions to make connections between the different representations to rewrite in equivalent forms (LP 3).
    - Reason about numerical relationships that can be rewritten using polynomial identities (LP 3).
    - Use multiple representations (table, graph, and symbols) to calculate the average rate of change (LP 3).
- use open-ended tasks to interpret data, analyze graphs, and reason about the context (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that polynomial and rational exponents, functions, and equations can authentically represent real-world situations (LP 6).

Often, coursework with polynomial and rational exponents, functions, and equations is focused on performing computations by hand. Instead, students should be given frequent opportunities to:

- use spreadsheets and graphing software to allow for focus on understanding, rather than computation or symbolic manipulation (LP 7).

Evidence of Learning

In M203 Polynomial and Rational Expressions, Functions, and Equations, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
203.a: Interpret the structure of polynomial and rational expressions.	i. interpret how parts of an expression, such as terms, factors, and coefficients relate to a real-world situation.
	ii. interpret complicated expressions by viewing one or more of their parts as a single entity.
	iii. use the structure of an expression to identify ways to rewrite it.
203.b: Reason about and perform operations on polynomials.	i. demonstrate understanding that polynomials are closed under the operations addition, subtraction, and multiplication.
	ii. add, subtract, and multiply polynomials.
203.c: Understand the relationship between the zeros and factors of polynomials.	i. identify zeros of polynomials when suitable factorizations are available.
	ii. use the zeros to construct a rough graph of a function defined by a polynomial.
203.d: Use polynomial identities to solve problems.	i. prove polynomial identities and use them to describe a numerical relationship.
203.e: Rewrite rational expressions.	i. rewrite simple rational expressions in different forms.
203.f: Interpret polynomial and rational functions that arise in applications in terms of the context.	i. interpret key features of graphs and tables in terms of the quantities for a function that models a relationship between two quantities.
	ii. sketch graphs showing key features given a verbal description of the relationship between quantities.
	iii. calculate and interpret the average rate of change of a

Content and Practice Expectations	Indicators Choose an artifact where you...
	polynomial or rational function over a specified interval.
	iv. relate the domain of a function to its graph, and where applicable, to the quantitative relationship it describes.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M203 Polynomial and Rational Expressions, Functions, and Equations (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a task like this:

Part 1: Make Sense

In the early 1600s, Johannes Kepler (1571–1630) studied the motions of the planets to find a good mathematical model for them. In 1619, he published his third law of planetary motion, which says how the orbital periods of the planets are related to their distances from the sun. In Kepler's time, Uranus and Neptune had not been discovered, but in the table below are data for all 8 planets:

planet	distance (millions of km)	period (days)
Mercury	57.9	88.0
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687.0
Jupiter	778.6	4,331
Saturn	1,433.5	10,747
Uranus	2,872.5	30,589
Neptune	4,495.1	59,800

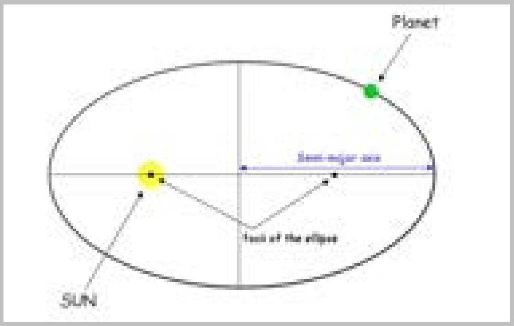
1. *What sort of function do you anticipate will fit this data? How did you arrive at that conclusion?*

Part 2: Graphing

2. *Plot the distance (x) and period (y) of each planet, and find a polynomial model that fits the data as well as possible. You may have to experiment with both the degree of your polynomial function and the number of terms.*
3. *Make another plot that uses the square root of each distance instead, and find a polynomial model that fits those points as well as possible.*
4. *Which model do you think is best?*
    a. *Jupiter has a lot of moons. Look up the periods and distances of some of them, and use that data to make a polynomial model of the relationship between period and distance for Jupiter's moons.*
    b. *Use your model to predict the period of another one of Jupiter's moons using the radius of its orbit.*
    c. *How good was the prediction? What are some possible sources of error?*

Part 3: Analyze

Kepler's Third Law of Planetary Motion states: "The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit".



[Keplers laws]

5. *What does Kepler's Third Law look like in terms of an algebraic equation? What do variables in that equation mean?*
6. *How can you use the information given in the table to determine the value of the constant?*
7. *Does the same constant hold for every planet?*
8. *How might this information be helpful in determining what polynomial function models this relationship?*

Part 4: Refine and Reflect

Jupiter has a lot of moons. Here are the periods and distances of the Galilean moons, which were discovered in 1610:

moon	distance (thousands of km)	period (days)
Io	421.8	1.77
Europa	671.1	3.55
Ganymede	1070.4	7.16
Callisto	1882.7	16.69

Note: Distance refers to the average distance from the center of Jupiter to each moon.

9. *Use the data to make a polynomial model of the relationship between a moon's orbital period and its distance from Jupiter's center.*
10. *Another moon of Jupiter, number LXXI, was discovered in 2018. Its distance from Jupiter is 11,483 thousand km. According to your model, what should its period be?*
11. *The actual period of the moon (LXXI) is 252.0 days. How close was the prediction? What are some possible sources of error?*
12. *What is something you learned about the work of developing functions that model real-world phenomena?*

Sample Learning Experience

Part 1: Make Sense

Launch this experience by inviting students to consider why studying the motions of the planets might be important. Consider using these questions:

- *What might the motion of the planets tell us?*
- *What factors do you think influence the motion of the planets?*

To deepen understanding on why studying planet motion is important, consider having students explore one or more of these resources:
Video: How the Movement of Other Planets Affects Earth — Yes, Really
Article: Planet Movements Impact Earth

Ask students:

- *What is something that you learned that was new?*
- *What surprised you?*

After discussing, present students the prompt and table above. Invite students to make sense of the table given:

- *What do you think the term "period" means here?*
- *What do you notice about the distance and period? Is there any relationship between the two?*
- *What type of association is represented in the table (linear, quadratic, power, etc.)?*

After discussing, let students know that we are working with this information to find a polynomial function that best fits the data. Give students time to consider what sort of function might fit this data and why, then ask them to share thoughts with their classmates, in pairs or small groups. Make note of students' ideas in a public space to refer to as the opportunity surfaces.

Part 2: Graphing

Ensure students have access to graphing technology, or share a link to the Desmos graph Planetary Distance and Period. As students work on this part, monitor and select students to share their approaches with the class.

Questions that can be posed to student pairs as they work:

- *What kind of growth is happening in the table? How do you know?*
- *What happened to your best fit equation as the degree changed?*
- *How would you explain the relationship between the period and distance of the moons?*

Consider using questions like this to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their work.

Part 3: Analyze

Invite students to share their models for the relationship between the orbital period and distance, and for the relationship between the orbital period and the square root of the distance.

Share with students the statement of Kepler's third law, which is: "The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit." This means that the square of the orbital period is the product of some constant and the cube of the distance. This can be expressed as $p^2 = n \cdot d^3$. This can be rearranged into something of the form $p = k(d^{1/2})^3$ which is the relationship they have been investigating. Once Kepler saw this relationship, he could divide the square of any planet's period by the cube of its distance in order to find out the proportionality constant.

The goal of this part of the journey is for students to explore these ideas for themselves and verify that all planets have the same proportionality constant. You may also invite students to compare what happens with Jupiter's moons—do they have the same proportionality constant as the planets? Newton's laws of gravity later explained why this relationship between distance and period is true. Give students to explore these ideas by conducting additional research.

Monitor and select students to share their approach as they work on these prompts. After some quiet time to think, encourage students to work with a partner or in small groups. Consider using the following questions to facilitate the discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their work.

Part 4: Refine and Reflect

Give selected pairs of students time to share responses and invite classmates to respond or pose questions.

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure and relationship between the equations to determine which sets of equations have the same answer. Give students additional time to revise their work.

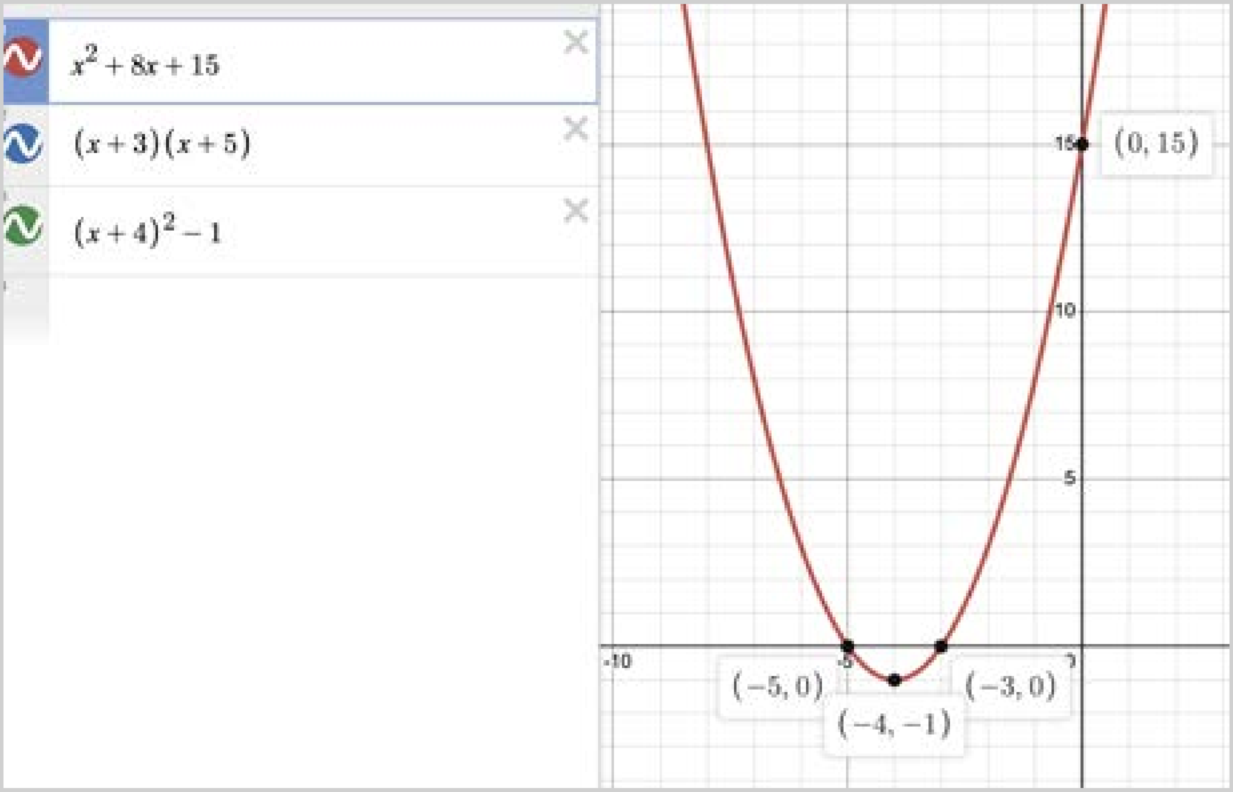
In this example, students are:

- given a cognitively demanding task. Students are asked to make conjectures about the relationship between a planet's distance from the sun and period. Students have multiple entry points to reason about this prompt in various ways (LP 1).
- using the varied structures that give opportunities to determine an approach by working independently, then sharing with a partner or class, and revising their response based on those interactions (LP 2).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces. Students build conceptual understanding as they work on their knowledge of polynomials, equations, and functions (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- developing a sense of identity and agency as they work to make sense of planetary motion and Kepler's Third Law, gaining insight into the work various career paths do when using mathematics to model relationships (LP 5 and LP6).
- leveraging technology to explore and determine the polynomial function to best fit the data given (LP 7).

Example 2

Students are given a task like this:

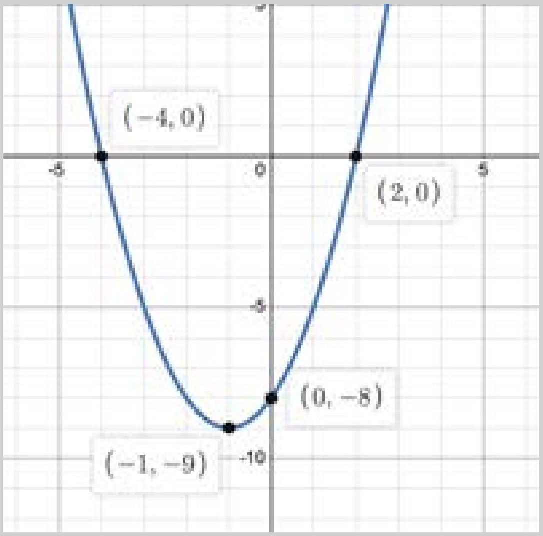
You have been given a polynomial graph with three equivalent equations. Analyze and make connections from the information in the questions below.



1. *Complete the chart below using the given graph and equations*

Write in each equation according to its form.	*Explain the features of the equation connecting the graph to this form.*	*When would using this form of the equation be most useful?*
Standard Form		
Vertex Form		
Factored		

2. *Now, you have been given the graph of a polynomial function, but the equations are missing.*



a. *Write an equation for the factored form. What do the values for x represent in this equation? Defend your thinking using mathematical and/or written justifications.*

b. *Explain the process for developing the vertex form of the function. What does the value for x represent in this equation?*

c. *Sketch the graph for your own polynomial equation using Desmos, linked here.*
d. *From the graph, write the three equivalent equations that would produce this graph.*
e. *Defend your thinking using mathematical and/or written justifications.*
f. *Summarize the importance of three forms of a polynomial function.*

Sample Learning Experience

Provide time for students to consider the question posed and independently develop a response. Have students write their response including their justification. Have partner pairs share their written justification and critique each other's response, allowing for students to defend their response. Provide time for students to revise their written justifications using the information gained during their partner-sharing time. While observing student discussions, identify varying solution methods to be presented during the whole group sharing. Have selected students present their responses—consider using a document camera for students to show actual work—and explain their justification.

As discussion takes place, invite classmates to join the discussion. Ask:

- *What about this reasoning makes sense?*
- *Is there anything you disagree with?*
- *What questions surface for you?*

Provide time for students to revise their work.

In this example, students are:

- given a cognitively demanding task. They are asked to analyze polynomial functions given points on the graph and corresponding numbers in the equation. Students have multiple entry points to reason about these equations in various ways (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking process and revise their own thinking (LP 2).
- working to make sense of the relationships between the standard, factored, and vertex forms of an equation. In so doing, students learn to recognize that different forms are helpful for identifying key information about the graph (LP 3).
- leveraging technology to examine a polynomial function and to support their conceptual understanding of intercepts, vertexes, and equivalent forms of an expression (LP 7).

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M204 Exponential and Logarithmic Functions and Equations

Badge Catalog Description How are exponents and logarithms related? Some quantities can vary by many orders of magnitude, like world population over time, housing prices, or the energy released by earthquakes. In M204 Exponential and Logarithmic Functions and Equations, you will gain the tools needed to understand and solve problems in situations like these. As you learn these mathematics, you will understand that exponential and logarithmic functions are inverse functions, gain the foundation to make sense of their graphs and characteristics, comprehend the rules for manipulating expressions involving exponents and logs to solve equations, and learn the relationship between base *e*, its exponential function, and the natural logarithm. Exponential and logarithmic functions are important nonlinear functions and are useful for careers in a variety of fields, like nuclear and internal medicine, forensic science, and finance. Suggested prerequisites for this badge: M103 Modeling with Functions of Quadratic Type; M104 Modeling with Functions of Exponential Type. This badge is suggested as a prerequisite for: Precalculus and Calculus.

The M204 Exponential and Logarithmic Functions and Equations badge puts reasoning and conceptual understanding at the center of how students engage with logarithmic and exponential functions. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with these functions. Work with exponential functions extends to include non-integer domains, while logarithms are developed from an informal understanding of “undoing” exponents.

As students engage in M204 Exponential and Logarithmic Functions and Equations, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M204 badge.

M204 Content and Practice Expectations

204.a	Understand the relationship between exponential and logarithmic functions.
204.b	Graph exponential and logarithmic functions.
204.c	Interpret logarithmic and exponential functions that arise in applications in terms of the context.

204.d	Use logarithms to analyze exponential models.

Learning Principles

In M204 Exponential and Logarithmic Functions and Equations, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting “doing mathematics,” but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M204

A typical instructional unit on exponential and logarithmic functions might focus on performing computations to complete tables and plot points in ways that are disconnected from real-world contexts. By contrast, in M204, students should:

- examine the structure of an exponential or logarithmic function before manipulating it in order to gain insights about the relationships involved (LP 3).
- use variables to represent quantities, defining exponential or logarithmic functions to represent mathematical or real-world situations (LP 3).
- recognize that different forms of exponential and logarithmic functions are helpful for different purposes or for particular contexts (LP 3).
- use logarithms as tools for solving exponential equations and employ reasoning to justify a solution or its reasonableness (LP 3).
- regularly encounter real-world tasks involving exponential or logarithmic relationships (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical (LP 3).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).

Whereas typically, students in a course on exponential and logarithmic functions may focus on memorizing and repeatedly applying “log rules” without understanding, in M204, students should:

- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, algebraic, numerical, graphical. As examples, students should engage with the following:
    - Reason about and explain the real-world meaning of parameters in an exponential or logarithmic function.
    - Make connections between the different representations of a situation.
    - Explain the relationship between parameters in an exponential or logarithmic function and the features of their graphs (LP 3).
- learn that logarithms are a way to express the exponent that satisfies an exponential equation (LP 3).
- use open-ended tasks to interpret data, analyze graphs, and reason about the context (LP 3).

- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that logarithmic and exponential functions can authentically reflect real-world situations (LP 6).

Often, coursework with exponents and logarithms is focused on manipulating symbols by hand. Instead, students should be given frequent opportunities to:

- use software to graph exponential and logarithmic functions (LP 7);
- use calculators and software to compute values for exponential and logarithmic expressions and interpret their contextual meaning (LP 7).

Evidence of Learning

In M204 Exponential and Logarithmic Functions and Equations, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
204.a: Understand the relationship between exponential and logarithmic functions.	i. use reasoning about exponents to explain the value of a logarithmic expression.
	ii. write an exponential expression to verify the value of a logarithmic expression.
204.b: Graph exponential and logarithmic functions.	i. graph an exponential function over a domain that includes non-integers.

Content and Practice Expectations	Indicators Choose an artifact where you...
	ii. graph a logarithmic function.
	iii. explain the effect of changing the parameters in an exponential or logarithmic function on its graph.
204.c: Interpret logarithmic and exponential functions that arise in applications in terms of the context.	i. interpret key features of the graph of an exponential function in terms of the context being modeled.
	ii. interpret key features of the graph of a logarithmic function in terms of the context being modeled.
	iii. explain the choice of domain for a logarithmic or exponential function in terms of the context being modeled.
	iv. calculate and explain the meaning of the average rate of change for an exponential or logarithmic function in terms of the context being modeled.
204.d: Use logarithms to analyze exponential models.	i. create an exponential model for a context.
	ii. use a logarithm to solve for an unknown value of an exponential model.
	iii. interpret values for exponential models in context.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M204 Exponential and Logarithmic Functions and Equations (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a prompt like this:

Consider the following scenarios.

1. *There are currently 289 million registered cars in the United States, increasing by about 1.5% annually.*
2. *For the original strand of COVID-19, one infected person was responsible for further infecting about two people during their infectious period. The spread of the virus continued in that pattern. The current US population is about 330 million people.*
3. *There are 20,000 fish in an infected pond. The population of fish declines to 8,000 in four hours.*

For each of the scenarios above, write down questions you could ask about the situation. Write as many questions as you can think of.

As a group, you will choose some interesting questions to answer. If you would need to do calculations, make a sketch, or draw a graph in order to answer a question, then it's probably an interesting question.

Write a report to explain your questions and answers. The report should include:

- *a context for your situation. Write a story that includes all the relevant information. If the description of the situation is missing some details, fill them in however you want.*
- *an equation to describe the situation, with an explanation of what each part of the equation represents.*
- *a labeled graph of the situation, including the questions you chose and your answers. Include your process for finding the answers. Say what the answers mean in the context of this situation.*

Adapted from Illustrative Mathematics Modeling Prompt (Illustrative Mathematics, 2019)

Sample Learning Experience

Provide time for students to consider the question posed and independently develop a response before they are encouraged to work with a partner or small group.

Monitor and select students to share their approach as they work on these prompts. Consider using the following questions to facilitate the discussion:

- *How would you describe the relationships captured in your preliminary work: linear, exponential, or something else? How can you tell?*
- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their approach. Give students additional time to explore their ideas and conduct additional research, as well as to prepare their report. Reports can take any form that students feel is appropriate: traditional presentation, TED Talk style, a video, an animation, or any other form that makes sense to them.

As students showcase their work, invite classmates to engage with the presentation. Consider asking the following questions:

- *What is something you appreciate about the presentation?*
- *What about this reasoning makes sense?*
- *Is there anything you disagree with?*
- *What questions surface for you?*

Provide time for students to revise their work.

In this example, students are:

- given a cognitively demanding task. Students are asked to generate questions of interest they want to explore. As they generate questions, they have the opportunity to explore possible relationships and representations, which sets the stage for the research component (LP 1).
- using the varied structures that give opportunities to determine an approach by working independently, sharing and collaborating with a partner or classmates. Students also have opportunities to refine and revise their response based on those interactions (LP 2).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces. As students consider the type of relationships they're formulating, they deepen their understanding of exponential relationships (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- developing a sense of identity and agency as they work to make sense of self-selected investigations, gaining insight into the work various career paths do when using mathematics to model relationships (LP 5 and LP 6).
- leveraging technology to explore and determine exponential equations or functions to best answer their questions (LP 7).

Example 2

Students are given a prompt like this:

The tuition at a college was $\$30,000\ in\ 2012$*, \$31,200 in 2013, and \$32,448 in 2014. The tuition has been increasing by the same percentage since the year 2000.*

1. *The equation* $c(t) = 30,000 \bullet (1.04)^t$ *represents the cost of tuition, in dollars, as a function of* t*, the number of years since 2012. Explain what the 30,000 and 1.04 tell us about this situation.*
2. *What is the percent increase in tuition from year to year?*
3. *What does* $c(3)$ *mean in this situation? Find its value and show your reasoning.*
4. *Write an expression to represent the cost of tuition in 2007. How much did tuition cost that year?*
5. *Typically, it takes students about five years to complete a bachelor's degree.*
    a. *If you were entering this college, what is the anticipated tuition for your first year?*
    b. *How much would you anticipate the total cost in tuition would be to complete your bachelor's degree?*
    c. *Select a career path that requires a bachelor's degree. Research how much someone in that career makes over time. Create an equation or graph of the potential earnings.*
    d. *Based on this work, how would you describe the impact of earning a bachelor's degree in that career path given the cost of tuition?*
    e. *Examine some other careers that don't require a bachelor's degree. Can you find any with similar earning potential?*

Adapted from Illustrative Mathematics Alg 2 Unit 4 Lesson 2 (Illustrative Mathematics, 2019)

Sample Learning Experience

After some individual work time where students develop an approach to the prompt, encourage them to share and compare approaches. Monitor and select a couple of approaches to highlight and reflect on progress made at this point. Look for student work that indicates students are beginning to attend to the structure of the equation. Make technological tools available so that students can reason using tables and graphs as they make sense of the structure of the exponential function given.

Questions that can be posed to student pairs as they work:

- *What do the numbers in the equation represent?*
- *What is the growth rate in the equation?*
- *How would you describe the change in tuition costs?*

Give selected pairs of students time to share responses and invite classmates to reflect on the following questions:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure and relationship between the equations to determine which sets of equations have the same answer. Give students additional time to revise their work. Repeat this process as students engage with Question 5.

In this example, students are:

- given a cognitively demanding task. They are asked to analyze components of an exponential function. Students have multiple entry points to reason about these equations in various ways (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking process and revise their own thinking (LP 2).
- attending to the structure of exponential functions and working to make sense of the relationships between the starting value, growth rate, and exponent. In so doing, students learn to recognize that different forms are helpful for different purposes (LP 3).
- encouraged to develop skills of expert learners, thereby developing agency, as they reflect on their approach to the prompt and incorporate new ways of thinking about this from their peers (LP 4).
- developing a sense of identity and agency as they work to make sense of costs of college and potential earnings to gain insights into the cost-benefits of a college education. In doing this work, students broaden their ideas about what mathematics is, how it is used, and who it is for (LP 5 and LP 6).
- leveraging technology to examine the nature of exponential functions (LP 7).

Example 3

Students are given a prompt like this:

When a virus invades our body, it gets our body to make so many copies of it that it disrupts normal functioning of our body at the cellular level and makes us sick. Watch this video from Life Noggin- Just How Fast Does a Virus Spread? to learn more. As the virus spreads to other cells in our body, the viral load *in our body increases. The* viral load *measures how much viral RNA is present in some body fluid, such as blood or saliva. The unit of measure is in copies per mL using* log_{10}.

Part 1: Making sense of log base 10

Let's explore how this works. Study the table below.

x	*10*	*100*	*1,000*
$log_{10}x$	*1*	*2*	*3*

1. *What do you notice? How are the numbers changing?*
2. *How would you describe what $log_{10}x$ is?*
3. *What would the graph of $y = log_{10}x$ look like?*
4. *What do you anticipate this has to do with the concept "viral load"?*

Part 2: Making connections to the medical field

A group of researchers set out to understand how viral load is associated with the risk of COVID-19 transmission. They found that the median viral load among the cases they looked at was "$5.6log_{10}$copies of RNA per mL of saliva." [source: [SARS-CoV-2 viral load is associated with risk of transmission to household and community contacts | BMC Infectious Diseases]().]

4. *What about this expression makes sense to you at this point?*
5. *What questions do you have?*

The medical field uses log_{10} both as a label and in its mathematical sense. Examine the table below to see how.

$Wlog_{10}$	*$1log_{10}$*	*$2log_{10}$*	*$3log_{10}$*	*$4log_{10}$*	*$5.6log_{10}$*
number of copies per mL	*10*	*100*	*1000*		

6. *Complete the table using patterns that you see.*
7. *The same researchers found that a small portion of cases had viral loads greater than $8log_{10}$. How many copies per mL is that? How much greater is this viral load than $4log_{10}$?*

Part 3: Reflecting on your learning

8. *What is something you've learned about the meaning of logarithms?*
9. *What do you think the significance of these numbers is? Why would anyone be interested in understanding the concept of viral load?*
10. *What is something you've learned about yourself, as you made sense of these ideas?*

Sample Learning Experience

Launch this experience by asking students what's the largest number they can imagine. You can use numbers they share to lead into the idea of how viruses make us sick, possibly asking: *do you think your body can hold that many copies of a virus?* Take a poll and make note of some of the ideas that surface in a visible space, like a poster or whiteboard. Reference these ideas as applicable throughout this journey. Listen for ideas that the Life Noggin video addresses as well as ideas that can transition to the concept of viral load.

Show the video and amplify the segment where the virus gets our cells to make multiple copies. Share with students that it is this phenomenon (replication of RNA) that experts in the medical field work to quantify with a unit of measure called *viral load.* Encourage students to share questions they have and, as time permits, allow students to research this topic.

Consider using these questions to debrief this segment:

- *What is something that you learned that was new?*
- *What surprised you?*

Part 1: Making sense of log base 10
Frame this learning experience by sharing that we will be working to understand how viral load is measured, and this entails very large numbers. Give students time to work on Part 1 individually, prior to inviting them to work with a partner or small groups.

As students work, consider posing these questions:
- *What do the values in the log function represent?*
- *How is the function growing? Where do we see that in the table? Graph?*
- *What connections are there between the different representations?*

Monitor and select varied approaches and responses to share out to the class. Consider using the following questions to facilitate discussion:
- *What about this makes sense to you?*
- *What questions does this work surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their work.

Part 2: Making connections to the medical field
Give students quiet work time before inviting students to partner or work with a small group. Consider pausing after Questions 4 and 5 to give space for students to share their current understandings and work to develop a shared understanding of the context. Capture ideas students share on a visible surface such as chart paper or white board.

Encourage students to engage in the next set of questions as a way to address some of the questions they've surfaced as well as to confirm some of the ideas they have.

As students work, pose the following questions:

- *What similarities do you see between the work you did in Part 1 and what you're looking at here?*
- *How can you use what you know from Part 1 to explain what you see happening in this table?*
- *What tools might be helpful to you: equation? table? graph?*
- *What connections are there between the different representations?*

Monitor and select students to share their thinking with the class. Use suggested questions as you see fit. Look for opportunities to connect to students' initial ideas and questions captured in early parts of this experience.

Part 3: Reflecting on your learning
Give students an opportunity to reflect on their learning and to share with others. As needed, provide time for students to revise and refine their work.

In this example, students are:
- given a cognitively demanding task. They have opportunities to use what they understand about exponentiation and multiplicative growth patterns to make sense of logarithms (LP 1).

- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions to their peers to understand one another's thinking process (LP 2).
- engaging in extending their understanding of exponentiation and multiplicative growth patterns to develop an understanding of logarithms(LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- given opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for as they examine the mathematical underpinnings of logarithms and their use in the medical field (LP 5).
- engaging with an authentic use of logarithms in the context of viral loads and examining why this matters (LP 6).
- leveraging technology to explore logarithmic expressions and functions as they develop conceptual understanding of the logarithmic functions (LP 7).

References

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M205 Trigonometric Functions

Badge Catalog Description Why do some real-world phenomena produce graphs shaped like waves? How can trigonometric functions be used to model the rising and falling of the ocean waters each day? Using radian measure, mathematicians have been able to define trigonometric functions, an intriguing function family with unique properties that can also help us to solve problems and model phenomena. In M205 Trigonometric Functions, you will explore the unit circle and develop understanding of how radians are defined. You will explore the properties of trigonometric functions, using graphing applications to aid in your analyses, and find opportunities to use these functions to model real-world situations. Trigonometric functions are useful for careers in a variety of fields, like the sciences, engineering, oceanography, and gaming. Suggested prerequisites for this badge: M155 Right Triangle Trigonometry.

The M205 Trigonometric Functions badge focuses first on developing students' conceptual understanding of the meaning of radian measure and why it is a useful alternative to degree measure. Students then explore the unit circle, using structure to make arguments for the values of sine, cosine, and tangent ratios for angles given in radians (MP7). Students similarly employ structural thinking and reasoning to understand the effects of different parameters on the graphs of trigonometric functions. Although M205 is not a modeling badge, emphasis is also placed on using trigonometric functions to model periodic phenomena, with the aid of graphing software.

As students engage in M205 Trigonometric Functions, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M205 badge.

M205 Content and Practice Expectations

205.a	Understand the radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
205.b	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers.
205.c	Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
205.d	Analyze trigonometric functions using different representations.
205.e	Use trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Learning Principles

In M205, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M205

A key component of M205 is an emphasis on developing a deep conceptual understanding of the meaning of radian angle measure. Often when students encounter radians, instructional time is focused on memorization; students memorize the definition of a radian and key conversions between radians and degrees (e.g., π is equivalent to 180 degrees). By contrast, in M205, students should view radians, the unit circle, and trigonometric functions as logical extensions of topics they have already encountered. In particular, they should:

- engage in discussion on the usefulness of radians as a vehicle for extending the domain of trigonometric functions to all real numbers (LP 3).
- use the structure of the unit circle to justify claims about the values of sine, cosine, and tangent for different angle measures, as well as the symmetry and periodicity of trigonometric functions (LP 3).
- focus their time on reasoning about the structure of the unit circle, rather than performing computations or memorizing (LP 3).

Student time can also be focused on procedural exercises that involve generating trigonometric functions that meet precise requirements (e.g., an amplitude of 7.5 and a phase shift of $\frac{\pi}{4}$). These exercises are usually devoid of any real-world contextual meaning and focus mostly on developing procedural skill with graphing and memorization of rules about the parameters of trigonometric functions. In M205 by contrast, students should be given opportunities to use a variety of representations to understand and analyze trigonometric functions. When possible, students should also use trigonometric functions in the context of realistic real-world applications and use technology as a tool for problem-solving. In particular, students should:

- use trigonometric functions as a problem-solving tool in authentic real-world situations, especially those that are related to authentic science and engineering disciplines (LP 1).
- explain the meaning of parameters for a trigonometric function that models a real-world situation (LP 3).
- engage with their peers to explain how an algebraic representation relates to a graphical representation of a trigonometric function (LP 2).
- engage in frequent experiences where they must make sense of given information, construct diagrams, test solution methods, and revise their thinking (LP 1).
- solve problems that shed light on social and/or environmental issues (LP 6).
- engage in learning experiences that emphasize reasoning and explanation over performing computations with trigonometric ratios (LP 2).

- employ a variety of tools, such as calculators and geometry software, to analyze and interpret trigonometric functions (LP 7).

Evidence of Learning

In M205, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts to present evidence of their learning related to the badge content and practice expectations.

Content and Practice Expectations	Indicators Choose an artifact where you...
205.a: Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.	i. use the definition of a radian to explain radian measure.
	ii. explain what an angle given with a measure in degrees would be in radians.
205.b: Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers.	i. explain how radians lead to a dimensionless number that makes it possible to extend the domain of trigonometric functions.
	ii. use the unit circle to justify the value of sine, cosine, or tangent for an angle that is greater than 90 degrees.
205.c: Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.	i. describe the symmetry of a trigonometric function.
	ii. use the unit circle to explain why a trigonometric function has symmetry.
	iii. describe the periodicity of a trigonometric function.
	iv. use the unit circle to explain why a trigonometric function is periodic.
205.d: Analyze trigonometric	i. create the graph of a trigonometric function.

Content and Practice Expectations	Indicators Choose an artifact where you…
functions using different representations.	ii. identify key features of the graph of a trigonometric function, such as its period, amplitude, and frequency, and connect these features back to the context of the problem.
	iii. explain the relationship between the parameters of a trigonometric function and its graph.
205.e: Use trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.	i. create a trigonometric function to model a real-world situation.
	ii. use a trigonometric function to answer a question about a real-world situation.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student’s artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student’s artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student’s artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M205 Trigonometric Functions (Optional)

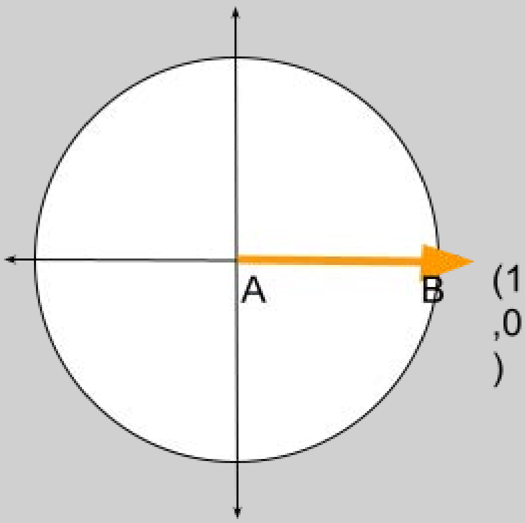
The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are presented with a problem like this:

Part 1
Draw a circle on a coordinate plane centered at (0,0) with a radius of 1. This is called a unit circle.

What is the circumference of the circle?



What is the length of the arc from point (1,0) to the point (-1,0)? How do you know?

Part 2
Use the following steps to find additional arc lengths, angles, and coordinates on the unit circle.

(1) *Find the arc length from point (1,0) to a different point on the unit circle.*
(2) *Mark and label the point J.*

(3) *Record the arc length in the table.*
(4) *Draw ray AJ.*
(5) *Determine the measure of the angle formed by ray AB and ray AJ. Record this in the table.*
(6) *What are the coordinates of point J? Record the coordinates in the table.*
(7) *Repeat steps 1-5 for 3 more points. Label them K, L and M.*

Point	Arc Length from (1,0)	Angle Measure	Coordinates
J			
K			
L			
M			

Part 3
Definition of 1 radian: An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.

Given the definition of 1 radian, approximately where would 1 radian be on your circle? Mark a point there and label it point P.

If you haven't already, explore the relationship between the sine and cosine of the angle measures from your table and the coordinates of the associated points. What do you notice? How can you explain the relationship that exists, if any?

Sample Learning Experience

The purpose of this task is to explicitly connect arc lengths on the unit circle to the radian as a unit of measure.

Begin by giving students quiet time to make sense of the information given and process Part 1 of the task they have to complete. Provide them with physical tools, such as graph paper, rulers, or digital geometry software to experiment with relationships made visible by the unit circle.

After some individual time, encourage students to discuss progress with a partner or small group. Encourage students to make their thinking visible. Initially, focus a conversation on how students interpreted and used the given information. Consider pulling the class together to discuss and share students' labels and use of tools to build shared understanding of the known information about the triangles. Monitor and select student work that aids in the conversation.

Ensure that students' understanding of Part 1 is secured before having them explore Part 2. Monitor for, select, and share student work that embodies different approaches and levels of formality for discussion. Listen for the various ways that students determine how to complete different columns in their table in Part 2. Did they use trigonometry or algebra knowledge to determine the coordinates? Did they estimate the angles and arc lengths or are they precisely determined?

As students share their understanding and reasoning with classmates, invite them to make sense of each other's work. Consider the following questions:

- *What are some things you appreciate about this explanation?*
- *What questions does this explanation surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work and share with each other again. Focus this second discussion on changes or adjustments that they have made based on their peers' work.

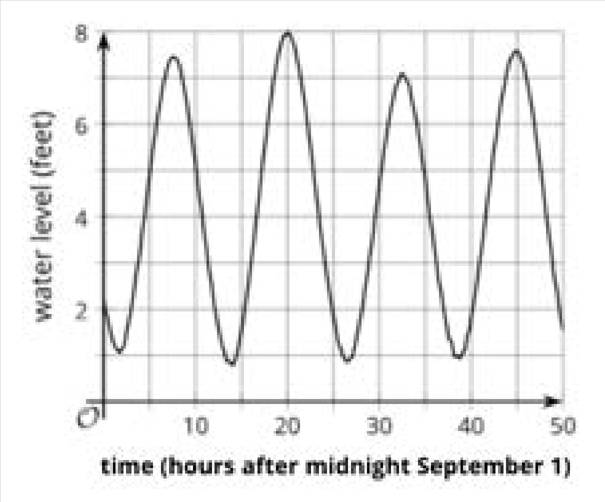
In this example, students are:

- given a cognitively demanding task. It requires students to reason using the structure of the unit circle to justify claims about arc lengths, angle measures, and coordinates on the unit circle. The learning experience is structured to give all students access by inviting them to write explanations with varying degrees of formality and giving an opportunity for revision after engaging with some of their peers' explanations (LP 1).
- working collaboratively through structured opportunities to ask questions to each other to better understand the thinking process (LP 2).
- building conceptual understanding through reasoning, as they must relate the unit circle and the work they did in right triangles in trigonometry (LP 3).
- encouraged to have agency, as they choose their solution pathways and are given space to reflect on how their approach might change based on engagement with their peers' work (LP 4).

Example 2
(**[Adapted from IM Alg 2 6.19 Summary]**)

Students are presented with a scenario like this:

Have you ever gone to the beach on a summer day and seen a yellow flag on the shore? This flag is an indication of moderate surf signaling that the tides may be high and swimmers should be cautious when entering the water. Oceanographers are people who research and study the different conditions of the ocean. One important job of oceanographers is to study the waves, currents, and tides. These reports on ocean activity help us to know when the ocean is safe and allowed to be open to the public for activities such as fishing, boating, and swimming.



1. *Above is an image of the water level in Bridgeport, Connecticut, over a 50 hour period in September. Using what you know about trigonometric functions, you will make sense of this information and do the job of an oceanographer for the day!*

    a. *What initial observations do you have about the water levels shown by the graph?*

    b. *Based on the image, what is the period for this graph? Explain how you know.*

    c. *Use Desmos to create an equation to model the relationship between the water level in feet and the number of hours after midnight on September 1st. What equation did you come up with? Explain how you created and refined your equation to get a close fit to the graph shown in this problem.*

d. *How might swimmers, boaters, or fishers use the information in your graph to make informed decisions about their activities? Is there anybody else who might be interested in the information shown by the graph? If so, who?*

Sample Learning Experience

Introduce the lesson by displaying the given image. Ask students: "*What do you notice? What do you wonder?*" Chart student responses close to the image, in a place that is visible to the whole class. Encourage them to elaborate in order to make the ways they are attending to structure visible to themselves and their classmates. Invite them to record some of these responses and elaborate as they examine question 1a.

Give students time to engage in the remaining questions individually, followed by partner or small group work. Monitor and select varied approaches to the prompts to share for whole class discussion.

As students share their approaches with classmates, invite them to make sense of each other's work, particularly when engaging with how they approached Question 1c. Consider the following questions:

- *What are some things you appreciate about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this explanation to apply to your own work?*

Following the sharing and discussion, give students an opportunity to revise their work or build upon their work to share with each other again.

In this example, students are:

- given a cognitively demanding task. They are asked to consider how to create a model of data that is periodic and use that information to make predictions (LP 1).
- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions of each other to better understand the thinking process, as well as given space to learn from their peers (LP 2).
- engaging in learning experiences that emphasize reasoning and explanation over performing computations with trigonometric ratios (LP 2).
- asked to explain the meaning of parameters for a trigonometric function that models a real-world situation (LP 3).
- invited to consider how the data related to tides can be used to make decisions about safety with different ocean activities (LP 6).
- employing a variety of tools, such as calculators and geometry software, to analyze and interpret trigonometric functions (LP 7).

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M211 Data Management and Visualization

Badge Catalog Description

How might a social media company's data about teen social media use differ from teens' self-reported hours of social media use? What part of a story is told by data? How is data generated or collected? What kind of measurement can give you insight into an unanswered question? Data can sometimes be very messy, and its collection requires ethical consideration.

In M211 Data Management and Visualization, you will explore the first steps a data scientist takes when dealing with univariate, bivariate, and multivariate data. You will organize data into rows and columns and learn how to deal with missing values. You will practice representing and describing data, transforming it as needed for a desired predictive modeling procedure. You will make decisions about what visual representation is best depending on the type of data you have. Using data visualization, you will consider what story you can tell from the data. After gathering data ethically, you can expect to use technology to organize, clean, and represent it in a meaningful way. Data management and visualization are useful for careers in a variety of fields, like data analytics, marketing, programming, and investigative journalism.

Suggested prerequisites for this badge: concepts of addition, subtraction, multiplication, and division; ratio concepts; solving problems involving percentages; M113 Modeling with Probability.

This badge is suggested as a prerequisite for: M212 Predictive Modeling; M213 Statistical Error and Predictive Model Validation.

The M211 Data Management and Visualization badge puts storytelling at the center of how students engage with gathering and organizing data. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding as they engage with real-life raw data and build on prior knowledge of one-variable measurement data to describe this data. Through creating data visualizations from large sets of data and repeatedly cleaning data to prepare it for predictive modeling, students gain fluency in the process of data preparation. Students build their adaptive reasoning through the aforementioned actions, as well as through describing important features of data visualizations in context. According to *Catalyzing Change in High School Mathematics*, "Data arise from a context and come in two traditional types: quantitative (continuous or discrete) and categorical. Technology can be used to "clean" and organize data, including very large data sets, into a useful and manageable structure—a first step in any analysis of data" (National Council of Teachers of Mathematics, 2017, p. 81).

As students engage in M211 Data Management and Visualization, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M211 badge.

M211 Content and Practice Expectations

211.a	Identify features that make some graphs of quantitative and categorical data misleading.
211.b	Identify the subjects and variables in a data set.
211.c	Interpret and compare graphical representations of quantitative data and categorical data.
211.d	Describe key features of raw quantitative data and categorical data.
211.e	Create graphical representations of quantitative data and categorical data.
211.f	Prepare raw data for predictive modeling by organizing, cleaning, summarizing, aggregating, and feature engineering.
211.g	Understand where data comes from and how to collect primary and secondary sources of data.
211.h	Perform data practices (collecting, generating, analyzing, and disseminating data) ethically, and evaluate data practices on their use of ethics.

Learning Principles

In M211 Data Management and Visualization, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home

lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M211

Often, coursework with data is focused on performing computations by hand or by using a handheld graphing calculator. Instead, students should be given continuous opportunities to:

- use spreadsheets to examine the nature of data before manipulating it in order to gain insights about the relationships involved, to organize and merge data sets to prepare them for predictive modeling, and apply spreadsheet formulas to help describe key features of data sets (LP 7).
- use tools within spreadsheets and coding languages to allow for a sharp focus on understanding summary statistics and key features of data, rather than computing statistics and drawing data visualizations by hand (LP 7).
- access online tools and resources such as databases, integrated development environments, videos, and data analysis platforms to allow for creating data visualizations, collecting data ethically, and summarizing key features of data (LP 7).

Students should also be given opportunities to understand and reflect on the ways that data management and visualization can authentically involve real-world situations (LP 6).

Whereas a typical instructional unit on data management might begin with a focus on students performing computations or manipulating data disconnected from real-world contexts, in M211, students should:

- examine the structure of data before manipulating it in order to gain insights about the relationships involved (LP 3).
- have opportunities to understand and explain why different data visualizations produce similar interpretations about their key features (LP 3).
- recognize that different forms of data visualizations are helpful for different purposes or for particular contexts (LP 3).
- regularly encounter real-world tasks involving data management (LP 1) that require them to make sense of multiple representations and how they relate to each other—verbal, algebraic, numerical, graphical (LP 3).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).

Specifically, in M211, students should:

- regularly engage with tasks that focus on understanding and reasoning about different representations—verbal, numerical, graphical. As examples, students should be asked to do the following:
  - Reason about and explain the real-world meaning of data visualizations.
  - Make connections between the different visual representations of a situation.
  - Engage with a variety of data sources and visualizations to reason about having similar or different key features of their graphs.
  - Hypothesize about relationships between variables by determining if there appears to be a positive, negative, or no association between the variables.
  - Determine the reasonableness of using certain variables and observations to make sense of data.
  - Interpret data visualizations in the context of a situation.
  - Identify, analyze, and synthesize relevant external resources to pose or solve problems.
  - Identify important quantities in a practical situation and map their relationships (LP 3).
  - Evaluate the ethics of a data collection process, as well as identify any bias.
- use open-ended tasks to interpret data, analyze graphs, and reason about the context (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments to include social interaction as part of the learning process (LP 2).

Evidence of Learning

In M211 Data Management and Visualization, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
211.a: Identify features that make some graphs of quantitative and categorical data misleading.	i. demonstrate understanding that axis scale, truncated axes, and pictographs can exaggerate differences between values.
211.b: Identify the subjects and variables in a data set.	i. given a data visualization, set of raw data, or study description, identify who or what the subjects are.
	ii. given a data visualization, set of raw data, or study description, identify what was collected or observed from the subjects.
	iii. demonstrate understanding in context of what the rows and columns represent in a set of raw data (i.e., rows represent subjects and columns represent variables measured or collected).
211.c: Interpret and compare graphical representations of quantitative data and categorical data.	i. summarize key features of graphs in context such as shape, center, outliers, variability, and direction of trends, if applicable.
	ii. compare key features of graphs in context.
211.d: Describe key features of raw quantitative data and categorical data.	i. calculate measures of center and measures of spread, and identify outliers and other summary statistics (such as maximum, minimum, and skew) of large data sets and explain their meanings.
211.e: Create graphical representations of quantitative data and categorical data.	i. select appropriate types of visual representations for the type of data.
	ii. use appropriate tools to create visual representations of large data sets.
211.f: Prepare raw data for predictive modeling by organizing, cleaning,	i. merge multiple data sources and explain why this is necessary.

Content and Practice Expectations	Indicators Choose an artifact where you...
summarizing, aggregating, and feature engineering.	ii. make decisions to remove data observations or fill in missing values of data and justify these decisions.
	iii. filter and/or sort data to identify key features of data.
	iv. aggregate data if necessary to create new variables for the modeling process and justify the need for these variables.
211.g: Understand where data comes from and how to collect primary and secondary sources of data.	i. distinguish between primary and secondary sources of data.
	ii. demonstrate understanding of types of sampling methods such as voluntary response, random sampling, stratified random sampling, systematic random sampling, and web scraping.
211.h: Perform data practices (collecting, generating, analyzing, and disseminating data) ethically, and evaluate data practices on their use of ethics.	i. define and apply terms surrounding data ethics, such as privacy, targeted advertising, location services, readability, and cookies.
	ii. demonstrate understanding of choosing sampling methods that responsibly prevent bias.

Annotated Examples M211 Data Management and Visualization (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1:

Students are given a prompt like this:

With your partner, you will write a business report for Me & the Bees. During this time, you will analyze data, synthesize your discoveries, and communicate your findings. You will use the tools and strategies you learned for working in a spreadsheet to continue to look for patterns in the data. You will then write up your findings in a business report. In your report to Me & the Bees, you will respond to the question, "What are your recommendations for Me & the Bees? Make sure to include evidence."

Your business report should include:

- *An introduction*
- *Discussion of findings with visual evidence as support*
    - *Visuals along with your analysis and synthesis of those visuals*
- *Recommendations and conclusions*
- *Business report writing features: concise, clear visuals, color, and graphs as supporting evidence*

Prepare to share your findings with the class.

Adapted from YouCubed.org

Note: Lessons from the YouCubed.org website may require setting up a free account and logging in to access the whole lesson.

Sample Learning Experience

To launch the lesson, ask the class if any of them are business owners or if they know any business owners. Share with them the idea that there is no age requirement to starting and running a business. Share with them the story of Me & the Bees through this video, and explain that companies like Me & the Bees will have a team in their company or an outside consulting firm analyze their data to give recommendations on how to improve their business. Let students know that they will be creating a business report for Me & the Bees. They will be given data to explore, model (find patterns within), and analyze. Finally, they will write a report to communicate their findings and describe recommendations for the company to better understand and improve their business.

In the week leading up to presentations, have class discussions using the following questions to guide their thinking:

- *What is the spreadsheet about?*
- *What does each row represent? What about each column?*
- *Brainstorm some things you could learn about the Me & The Bees business from the data in the spreadsheet.*
- *What are some questions you still have about it?*
- *What are additional things about the business you want to know?*
- *What steps would you need to take to merge the data into one spreadsheet?*

Give students some individual work time to explore the Me and the Bees Spreadsheet. Share this video to assist students in cleaning and preparing this data for their reports. Then have students pair with a partner to share their approach to this investigation. Provide partner pairs access to tools (such as chart paper, Google Slides, Sheets, Docs) to create a display of their findings. As students work, circulate and identify approaches to present to the whole class.

Be mindful to emphasize different approaches to providing visual evidence for both univariate and bivariate data (bar charts, pie charts, boxplots, etc.). Encourage students to ask questions of each other as solutions are presented.

In Example 1, students are:

- given a cognitively demanding task. Students are given an open-ended prompt to discuss findings within a set of data; in doing so, students have an opportunity to construct a viable argument. Students have multiple entry points to reason about this prompt in various ways (LP 1).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces (LP 3).
- engaging in a real-world task and presenting findings in a way that maps to a wide variety of future potential careers via a business report (LP 6).
- using technology to organize and make sense of data to more easily make recommendations (LP 7).

Example 2:

Provide students with American Community Survey Data and a prompt as follows:

In this activity we will collect bivariate data to see if there is a relationship between two variables: number of gallons of water used and how many people are in a household.

After a few days or a weekend passes to allow students to collect water use data, resume the activity:

Your group will use the Colab program, Box Plots – Explore Your Own Data, to tell the story of the variable(s) from the perspective of your local data. Then you'll explore the similarities and differences in your local data set from your survey and the state data set from ACS (American Community Survey Data). For each variable, your group will consider the reasons why the similarities and differences might *exist and if they point to the same need.*

Adapted from YouCubed.org

Sample Learning Experience

In pairs or groups, ask students to brainstorm a list of the different ways that water is used in their homes or living spaces. Share this as a class and generate a class list. Given this list, ask the class what they would estimate as their daily water usage in their household or living space in gallons of water per day. In their groups, students can make an estimate of household water usage based on one group member's data who is comfortable sharing with the group, as well as this group member's number of people within the household or living space. Have groups share their estimates with the class and record them.

Students are collecting this bivariate data to see if there is a relationship between the two variables: number of gallons used and how many people are in a household or living space. You can have a conversation with the class. Ask them:

- *Do you think there will be a relationship between these two variables?*
- *Why or why not? What relationship would you expect?*
- *What other factors might play into water usage?*

Let them know that this is an example of bivariate data. Give students a few days or a weekend to estimate their household water usage. Teachers may copy this form to collect data from students.

After some individual work time on the prompt where students develop an approach, encourage them to share and compare approaches. Look for student work that provides very different viewpoints about the data to bring up during the class discussion.

Bring the class together for a whole group discussion on what they noticed about the comparison between survey and state data. Invite students to share how the local data gave them new perspectives on their variable(s) and need or how there was no change at all. Ask them to think about why in some cases the data was very similar and in other cases there was a difference in the local and state level story.

Consider using this discussion to encourage wonderings about who is represented in each data set and how the difference we see in the local- and state-level data is a result of who was surveyed. Something to keep in mind is that not only does it matter who was in the survey but also the population the survey is meant to represent. This conversation will encourage students to be curious about the idea and ethics of sampling which they explore next, improving upon their answers from this activity to write a letter to their local state government.

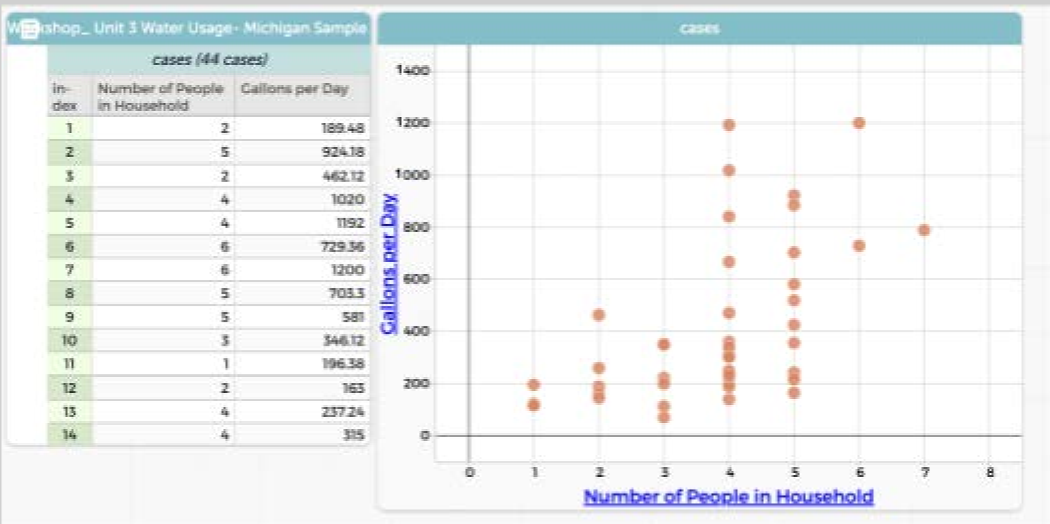
In Example 2, students are:

- working collaboratively to compare answers to the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking process and revise their own thinking (LP 2).

- encouraged to develop skills of expert learners, thereby developing agency, as they reflect on their approach to the prompt and incorporate new ways of thinking about this from their peers (LP 4).
- thinking about mathematics as a human endeavor as they recognize that the data includes the community in which they live (LP 5).
- using technology in order to visualize and compare multiple data sets (LP 7).

Example 3:

Provide students with the following prompt:



index	Number of People in Household	Gallons per Day
1	2	189.48
2	5	924.18
3	2	462.12
4	4	1020
5	4	1192
6	6	729.36
7	6	1200
8	5	703.3
9	5	581
10	3	346.12
11	1	196.38
12	2	163
13	4	237.24
14	4	315

Your group will explore water usage data in CODAP to model and analyze the relationship between the number of people in a household and amount of water usage. CODAP is the modeling tool your group will use to describe trends in bivariate data visualizations to tell the story of the relationship between people and water usage. Be prepared to share the following with the class:

- *A description of the relationship between the number of people in a household and amount of water used.*
- *A description of how you would choose to place a line and what story it tells you about the data. What were your considerations? Why do you think your line is a good representation of the data?*

Adapted from YouCubed.org

Sample Learning Experience

Launch the lesson with a discussion on the data visualization of the number of people in a household and the gallons of water used. Ask students to refer to the graph of the data shown above. Use the following questions to facilitate conversation:

- *What does the data look like? What do you notice?*
- *According to the data visualization, does there appear to be a relationship between people and the amount of water used in a household? Interpret the relationship or lack of relationship in*

context.

Encourage students to create a line that they think best represents their data applying the 'Movable Line' feature in CODAP.

After some individual work time, where students develop an approach to the prompt, encourage them to share and compare approaches. Monitor and select a couple of approaches to highlight and reflect on progress made at this point. Look for student work that indicates students are attending to context, shape, direction, strength, and outliers.

In Example 3, students are:

- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to consider discussion questions with their peers to understand one another's thinking process (LP 2).
- invited to give a context to the relationship between two variables to support their ability to communicate their conceptual understanding of the meaning of their interpretation instead of focusing on the procedural steps for solving. Students also work to reason about this relationship in context, making way for a deeper understanding of linear relationships (LP 3).
- leveraging technology to examine a relationship between variables and to support their conceptual understanding of interpreting linear relationships (LP 7).

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M212 Predictive Modeling

Badge Catalog Description

Can the number of people in a household be used to predict water usage? Can you use technology to predict whether or not you will like a song before hearing it? Can the amount of caffeine consumed predict pulse rates amongst your classmates? Are there meaningful categories in your data that technology can detect?

In M212 Predictive Modeling, you will master the skill of asking statistical questions, deciding what kind of model will answer your question, and a few types of supervised and one or two types of unsupervised learning methods that will hopefully help you answer your questions. Supervised learning methods rely on the idea that an output can be predicted from one or multiple inputs with some amount of error. Such methods may include Linear Regression, Multiple Regression, Logistic Regression, Classification, or Decision Trees. Unsupervised learning methods allow technology to recognize patterns in the data or categorize the data to make sense of it. Such methods may include Cluster Analysis, Dimensionality Reduction, and Feature Selection. You will distinguish between correlation and causation when a relationship appears to exist between variables. Predictive modeling is useful for careers in a variety of fields, like actuarial science, engineering, public health, and environmental sciences.

Suggested prerequisites for this badge: M211 Data Management and Visualization.

This badge is suggested as a prerequisite for: M214 Statistical Error and Predictive Model Validation.

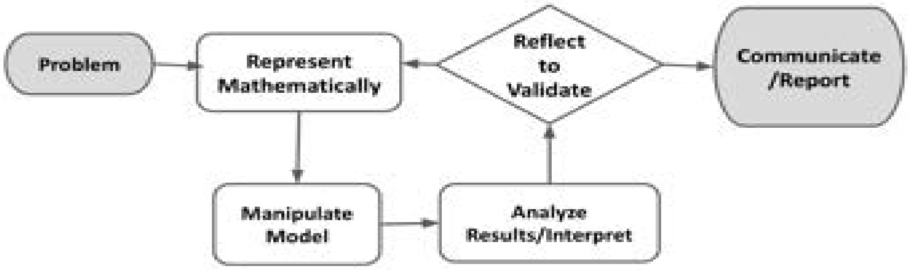
The M212 Predictive Modeling badge integrates mathematical modeling as an essential component of how students engage with making and interpreting predictions that require either supervised or unsupervised learning methods. In previous experiences in mathematics, students have dabbled in making predictions using supervised learning methods, where the input and output of their models were clear.

In M212, students may identify a strong, linear relationship between variables such as shark attacks and ice cream sales, but it is important for them to recognize contextually that differentiating between input and output, otherwise known as explanatory and response variables, does not really make sense. They will recognize that temperature likely has an influence on both of those variables, which will not only illustrate the important difference between correlation and causation, but it will also lead students to think about what variables they can control for if they conduct studies themselves.

Students will also likely see unsupervised learning methods for the first time, where they trust technology to make sense of their data in order to infer underlying patterns without using a specified input or output. This allows for content design built on relevant and authentic tasks that integrate concepts and skills acquisition with modeling, allowing for a coherent experience for students.

According to *Catalyzing Change in High School Mathematics*, "The focus of data analysis is on the underlying structure of the data as a whole and how the analysis might ... lay a foundation for predicting how events might unfold in the future" (National Council of the Teachers of Mathematics, 2017, p. 77).

The learning expectations for M212 Predictive Modeling center on the CCSSM modeling cycle as described here:



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

(Adapted from Common Core Standards Writing Team, 2019, p.6)

> A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. ... Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. ... The basic modeling cycle is summarized in the diagram.
> It involves
> 1. Identifying variables in the situation and selecting those that represent essential features.
> 2. Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
> 3. Analyzing and performing operations on these relationships to draw conclusions.
> 4. Interpreting the results of the mathematics in terms of the original situation.
> 5. Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> 6. Reporting on the conclusions and the reasoning behind them.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010 p. 72).

As students engage in modeling with supervised and unsupervised learning methods, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M212 badge.

M212 Content and Practice Expectations

212.a	Engage in the modeling cycle.
212.b	Construct questions that can be answered by predictive modeling.
212.c	Select appropriate types of predictive models that help answer statistical questions and justify these choices.
212.d	Demonstrate understanding that Data = Model + Error, where the Model may contain one predictor, multiple predictors, or no predictors at all.
212.e	Understand and implement machine learning methods, and distinguish between supervised and unsupervised learning methods.
212.f	Improve predictive model fit by transforming variables, selecting variables, or using an alternate type of predictive modeling method.
212.g	Use predictive models to make predictions and interpret these predictions in context.
212.h	Distinguish between correlation and causation.

Learning Principles

In M212 Predictive Modeling, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M212

Often, coursework with data is focused on performing computations by hand or by using a handheld graphing calculator. Instead, students should be given continuous access to:

- tools within spreadsheets and coding languages that allow for a sharp focus on understanding how to fit models to data, rather than computing models by hand (LP 7).
- online tools and resources such as databases, integrated development environments, videos, and data analysis platforms to allow for model creation (LP 7).

Students should also be given opportunities to understand and reflect on the ways that predictive modeling authentically involves real-world situations (LP 6).

Whereas a typical instructional unit on modeling might begin with a focus on students independently performing computations or manipulating various functions disconnected from real-world contexts, in M212, students should:

- regularly encounter real-world tasks involving a need for predictions that require them to employ models for these predictions (LP 1).
- engage with multiple parts of the modeling cycle (see above), especially naming their own assumptions and variables and defending their choice of model as much as possible (LP 1).
- be able to choose tasks that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate and share their solution methods (LP 2).

Spending time computing or manually completing tables with predictive models should not be a focus of this badge. In fact, the opposite is true. In M212, students should:

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with models. For example, students should:
    - explain the real-world meaning of a predictive model.
    - compare two or more predictive models.
    - explain the relationship between parameters in a predictive model and features of its graph, if applicable.
    - use a predictive model to justify a claim about a real-world context.
    - identify and explain circumstances where making predictions would be considered extrapolations and might not be a trusted prediction.
- have frequent opportunities to use reasoning to relate the algebraic form of a predictive model to its graph (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Evidence of Learning

In M212 Predictive Modeling, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence that includes at least one Performance Assessment that demonstrates successful engagement with the entire modeling cycle
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process.

Content and Practice Expectations	Indicators Choose an artifact where you...
212.a: Engage in the modeling cycle. Note: Satisfactory completion of an associated performance assessment fulfills this portfolio requirement.	i. engage with the full modeling cycle (problem, formulate, compute, interpret, validate, revise as necessary, report).
212.b: Construct questions that can be answered by predictive modeling.	i. differentiate between statistical and non-statistical questions by recognizing that a statistical question can be answered by collecting data and anticipates variability in those data.
	ii. formulate statistical questions with the intent of answering the question using a predictive model.
	iii. postulate statistical questions that produce a given data visualization/model to answer those questions.
	iv. identify and define independent, dependent, and codependent relationships between potential variables in a given context or data set.
212.c: Select appropriate types of predictive models that help answer statistical questions and justify these choices.	i. identify the most appropriate model to use based on the objective of the statistical question, how the variables are measured, and type of data used (e.g., group versus regression models).
	ii. informally assess model fit visually and by comparing the coefficient of determination when applicable.
212.d: Demonstrate understanding that Data = Model + Error, where the Model may contain one predictor, multiple predictors, or no predictors at all.	i. define and use the concept of a statistical model as a function that produces a predicted score for each observation.
	ii. differentiate between the empty model and a more complex model.
	iii. demonstrate understanding of how to represent a linear relationship with a Least Squares Regression Line and interpret the slope and y-intercept in context.
	iv. understand that a perfect prediction does not usually exist by recognizing error visually.
212.e: Understand and implement machine learning methods, and	i. build generalized linear models to fit data with one or more predictors, some of which may be transformed, and use those

Content and Practice Expectations	Indicators Choose an artifact where you...
distinguish between supervised and unsupervised learning methods.	models to predict outcomes.
	ii. demonstrate understanding of which classification methods are used by implementing at least one type of classification method to make a prediction.
	iii. demonstrate understanding of which classification methods are used by implementing at least one type of clustering method to make a prediction.
212.f: Improve predictive model fit by transforming variables, selecting variables, or using an alternate type of predictive modeling method.	i. demonstrate understanding that some models visually appear to fit data better than other models, which can be confirmed by an increased coefficient of determination.
	ii. use dimensionality reduction techniques to decrease the number of variables in a predictive model.
	iii. produce a predictive model that has transformed variables.
212.g: Use predictive models to make predictions and interpret these predictions in context.	i. appropriately plug inputs into a predictive model to obtain an output, called a prediction.
	ii. explain what an output from a predictive model means in context, including the appropriate use of units when applicable.
	iii. identify limitations of a predictive model, including the trustworthiness of extrapolations and evaluating intercepts/slope in context, when applicable.
212.h: Distinguish between correlation and causation.	i. demonstrate understanding that correlated variables do not imply a causal relationship between those variables.
	ii. define confounding variables and describe the contextual meaning of an identified confounding variable.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution	The student's artifact demonstrates evidence that they have met the

instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	expectations of the indicator(s).

Annotated Examples M212 Predictive Modeling (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Sample Learning Experience

In groups, ask students to create a visual representation that communicates the relationship between the following terms: correlated, spurious correlation, causation, random relationship, direct relationship, confounding variable, and mediating variable. Some examples here.

As you walk around groups, encourage students to share their thinking with each other as they decipher between these terms and decide on a visual that they feel represents the situation. The process and conversation is more important than the final product.

After groups have created a visual, invite them to share their visual with the class. Highlight the many different ways that groups have represented the relationship. As a class, discuss the different representations and the meanings of the terms, considering how they relate to each other.

Adapted from YouCubed.org

Note: Lessons from the YouCubed.org website may require setting up a free account and logging in to access the whole lesson.

In this example, students are:

- given an open-ended task that allows students to choose a scenario or representation of their own understanding of crucial vocabulary terms. This non-routine task lays a foundation for students to more fluidly understand predictive relationships when they start making their predictive models (LP 1).
- set up to have social interactions, as they work in groups to model understanding of vocabulary terms used in predictive relationships (LP 2).
- encouraged to have agency, as they choose and reflect on and refine their approaches. We can further increase students' agency in learning by offering a more open task (LP 4).
- given the opportunity to reason and justify their self-made visual representations and listen to their classmates' reasoning and justifications of their visual representations (LP 3).

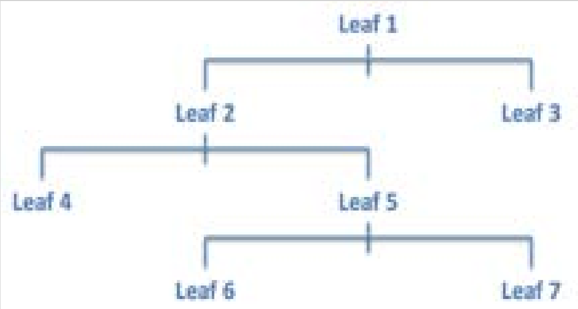
Example 2

Students are given a prompt like this:

A decision tree is similar to a series of questions that are asked sequentially. Observations start by answering the first question (at the root of the tree), and then proceed along the different branches based on the answers they give to the questions that follow. At the end, based on all of the questions asked, observations are then classified as one of k classifications.

*Your goal today is to work with your group to design a decision tree that may help solve a classification problem. Your decision tree should have at least seven clearly defined **nodes**/leaves that either classify a person/object or prompt the need for a new leaf. Draw a diagram of your group's decision tree.*

Example decision tree structure:



Once your decision tree is complete, pair with a group to test out the effectiveness of each group's decision tree. Ask your partner group to think of a person/object that can be classified into one of your group's categories, and have them move down your decision tree until their person/object is classified. Identify if the person/object was properly classified, then take turns attempting to use your partner group's decision tree. Provide feedback on whether or not your person/object is properly classified.

Make updates to your decision tree based on your partner group's feedback and prepare to have a whole class discussion.

Sample Learning Experience

As students work on their decision trees, ask questions that help reinforce students' confidence in their choice of categories. Provide examples if students struggle to start (e.g., types of fruit, baseball player or basketball player, school subject, etc.).

Adapted from [Introduction to Data Science Curriculum]

After groups make final modifications to their decision trees, discuss as a class:

- *How do decision trees classify objects/people as being a member of a group?*
- *Did our decision trees make correct predictions every time?*
- *How can we figure out what questions to ask and in what order to minimize the number of misclassifications?*
- *How is a decision tree similar to or different from a linear model?*

- *Can we call a decision tree a predictive model? Why or why not?*

In this example students are:

- grappling with the non-routine task of creating a decision tree. Even if multiple groups choose the same classification ideas, there are multiple solution pathways for every idea (LP 1).
- conducting this assignment socially to allow for multiple classification ideas that could work for a small-scale decision tree. After discussing and designing this tree as a group, students then receive feedback from another group as to whether or not a new object/person is properly classified. Students use this feedback from their social experience to revise their work (LP 2).
- able to choose a classification problem that interests them most, providing agency over the topic they decide to use (LP 4).
- doing math that, by nature, involves authentic real-world situations. While students choose the classification problem they are most interested in designing a decision tree for, the purpose of these trees is to classify a person or an object (LP 6).

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M213 Bayesian Reasoning and Probability Theory

Badge Catalog Description

Does the probability of becoming a professional athlete differ based on height? Does the probability of seeing yourself represented in media differ based on skin tone? Does the probability of flipping "tails" using a fair coin decrease if you just flipped "tails" three times in a row? Outcomes may differ depending on what events have occurred prior, which means that the probabilities within a sample space could be different from what they would be without these prior events.

In M213 Bayesian Reasoning and Probability Theory, you will learn the meaning of independence and how it relates to Bayes' Rule by learning and using multiple other probability rules. The use of two-way tables will help break down some of these concepts. You will learn that probability, by definition, is the expected proportion of times a desired event occurs. Using two-way tables and probability rules, you will calculate probabilities and decide if events are independent. You will recognize that these probabilities may change depending on prior conditions. You will strategically use tools like tables, tree diagrams, counting techniques, and the rules of probability. Using technology, you will simulate random processes, approximate probabilities, and interpret results. Bayesian reasoning and probability theory are useful for careers in a variety of fields, like biology, epidemiology, and risk management.

Suggested prerequisites for this badge: M211 Data Management and Visualization.

This badge is suggested as a prerequisite for: M214 Statistical Error and Predictive Model Validation.

The M213 Bayesian Reasoning and Probability Theory badge puts understanding of conditional probabilities at the center of how students engage with real-world scenarios and multiple kinds of probability distributions. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving by using inductive reasoning about prior events as they engage with probabilistic scenarios in the field of Data Science.

According to *Catalyzing Change for High School Mathematics,* "Probability theory brings the power of mathematics to describe randomness and chance. ... Experience with and knowledge about probability challenges intuitive ideas about chance and enables students to become members of society who have the capacity to make reasoned decisions about uncertain outcomes" (National Council of Teachers of Mathematics, 2017, p. 87). As students use their foundational knowledge of probability to reason inductively about probabilistic scenarios, students gain insights into the mathematical consequences of making assumptions about prior events and deepen their understanding of probability and its place in the world.

As students engage in M213 Bayesian Reasoning and Probability Theory, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M213 badge.

M213 Content and Practice Expectations

213.a	Understand that probability is a long run relative frequency.
213.b	Determine probabilities of single and multiple events using Bayesian reasoning.
213.c	Reason about the mathematical consequences of certain prior events.
213.d	Make likelihood predictions for discrete and continuous probability distributions.
213.e	Understand how sampling distributions, developed through simulation, are used to make likelihood predictions.

Learning Principles

In M213 Bayesian Reasoning and Probability Theory, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M213

In M213, students should be given frequent opportunities to use technology to simulate chance experiments and sampling methods to increase understanding (LP 7).

Whereas a typical instructional unit on probability might involve fabricated, artificial scenarios, in M213, students should be given opportunities to reason with probability in ways that are relevant to the field of Data Science. In M213, students should:

- recognize that different types of probability distributions or approximations to probability distributions are helpful for different purposes or for particular contexts (LP 3).
- mathematically claim and employ reasoning to justify a solution or its reasonableness (LP 3).
- regularly encounter real-world tasks involving discrete or continuous distributions that require them to compute probabilities to make sense of the plausibility of a prediction (LP 1).
- be able to choose tasks or entire courses that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- frequently collaborate, share their solution methods, and make their thinking visible (LP 2).

Specifically, in M213, students should:

- regularly employ previously learned probability concepts and skills, namely, designing simulations, using models to determine probabilities, computing probabilities of compound events, interpreting probabilities in context, and understanding independence and conditional probability. The focus of M213 is to build on these skills and apply them to the field of Data Science.

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with sampling distributions and various types of probability distributions. For example, students should:
    - understand probabilities as the long-run relative frequency of an event;
    - relate probability models to simulations and sampling distributions;
    - explain the real-world meaning of probabilities;
    - compare probabilities;
    - consider prior events and knowledge that may affect a probability;
    - use a probabilistic model to justify a claim about a real-world context;
    - use probability to make decisions (LP 3).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand and reflect on the ways that probability, Bayesian reasoning, and sampling distributions can authentically involve real-world situations (LP 6).

Evidence of Learning

In M213 Bayesian Reasoning and Probability Theory, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	**Indicators** **Choose an artifact where you...**
213.a: Understand that probability is a long run relative frequency.	i. describe probability contextually as a long run relative frequency and understand probabilistic implications for individual trials.

Content and Practice Expectations	Indicators Choose an artifact where you...
	ii. demonstrate understanding that probability is needed in the field of Data Science in order to provide guiding principles for obtaining information from data.
213.b: Determine probabilities of single and multiple events using Bayesian reasoning.	i. determine intersection probabilities using the conditional probability formula.
	ii. show and explain the correspondences across representations of compound probabilities (e.g., tree diagrams, two-way tables).
213.c: Reason about the mathematical consequences of certain prior events.	i. use Bayesian reasoning to compare the compound probabilities of independent and dependent events.
	ii. use understanding of the conditional probability formula, Bayes' Theorem, to describe the implications of prior events when making predictions.
	iii. demonstrate understanding that an association that exists between variables may disappear or reverse when observing subgroups.
	iv. describe a situation where independence may be assumed, even when drawing without replacement.
213.d: Make likelihood predictions for discrete and continuous probability distributions.	i. make likelihood predictions for at least one type of discrete probability distribution.
	ii. make likelihood predictions for a normal distribution.
	iii. check and describe conditions to approximate a distribution to a normal curve in order to make likelihood predictions.
213.e: Understand how sampling distributions, developed through simulation, are used to make likelihood predictions.	i. develop simulations to determine approximate sampling distributions.
	ii. explain how sampling distributions, developed through simulation, are used to describe the sample-to-sample variability of sample statistics.
	iii. make likelihood predictions using sampling distributions.

Criteria for Success:

Conference and Provide Revision Support	**Accept with Revision**	**Accept**
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M213 Bayesian Reasoning and Probability Theory (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Sample Learning Experience

Explain to students that they are going to create a class playlist and identify the genre categories they are in. The goal is to represent four-six different genres in the class playlist in order to make the probabilities more meaningful and to create a program that isn't too unruly. Have students and a partner write a list of 10 of their favorite songs, including the title and artist, then identify the genre category for each song.

Once pairs have their lists of songs and genres, bring the class together. Ask them to share the genres in their list. Consider organizing this like a whip around with new ideas only: after the first pair shares their genres, have each pair share only new genres to add to the list. If there are more or less than four-six genres generated by the class, start a conversation with them about needing to consolidate or expand the list. Have pairs input their final 10 song choices into a Google Form (it can look like this). Generate the spreadsheet from the form and share a copy with the students.

Students should clean the data by checking for typos or duplicates, then download the file as a .csv file. Provide this EduBlocks program for them to run their file in. Explain that this is a simulation to model listening to a playlist on shuffle, and that they are going to take some time now to analyze how realistic the model might be. Invite students to share how they think the simulation could more accurately model listening to a playlist on shuffle.

Building on what students share, emphasize that the programs they built had some limitations because they assumed replacement of songs back into the playlist. Share what this means about how songs are played when listening to the playlist. Point out that most playlist shuffles don't work in this way. Invite students for a short brainstorm about the question:

- *How would we need to change the program to model the way most people listen to playlists with each song playing once?*

Continuing the discussion, invite students to share what they think about how the related probabilities in the simulation would change or not without replacement. Ask questions about the probabilities like:

- *Now that you have talked about how the code would change, are there changes in the theoretical probability?*
- *Will the experimental probability be different without replacement? What if you play exactly the number of songs in the playlist?*
- *How does the variability of the simulation with replacement compare to that of the simulation without replacement?*
- *What questions are you interested in exploring with a simulation without replacement?* Record three-five questions students brainstorm.

Share with students that they will be exploring conditional probability questions about their model without replacement. For example: What is the probability of a rock song playing second if the first song was a rock song? Ask the students:

- *Are any of the questions the class generated conditional probability questions?*

Allow students time to revise the provided program in groups in order to reflect the questions generated and outcomes of the class discussion. Have groups swap programs and test them to see what similarities and differences they have in code. Then, have students revise their code one last time after exploring another group's.

Adapted from YouCubed

Note: Lessons from the YouCubed.org website may require setting up a free account and logging in to access the whole lesson.

In Example 1, students are:

- given a cognitively demanding task. They are asked to explore and explain conditional probability as well as how it connects to a real-life context (LP 1).
- using the varied structures that give opportunities to determine an approach by working in pairs, then sharing with another group or class, and revising their response based on those interactions (LP 2).
- given a task that centers reasoning with multiple opportunities to make sense of the impact of drawing without replacement. Students build in-depth understanding as they work to justify these changes and make their program make sense in context (LP 3).
- leveraging technology to examine whether a provided simulation reflects a real-life version of a song shuffler and to support their conceptual understanding of conditional probability (LP 7).

Example 2

Students are shown the following photo and prompt:

- *What do you notice about the usage of the weights based on the wear of the weights (where the pin has been placed)?*
- *What do you wonder?*
- *What story does this visual tell?*

Sample Learning Experience

Invite students to share their noticings, wonderings, and questions. As a class, discuss:

- *What do the marks on the weights tell us about gym goers' use of weights?*
- *If we were to make a histogram of the use of different weights, what do you think it would look like?*

This data talk depicts how weight usage (as marked by the wear on the weights created by users missing inserting the pin, indicating higher usage) falls along a normal distribution. It is interesting to note that by looking at the wear and tear based on the number of pin misses, we are able to see a normal distribution curve.

In groups, ask students to sketch a histogram that represents weights usage by gym goers. Students should then discuss the following questions with each other and take notes on their findings:

- *What do you notice about the shape of the histogram?*
- *What are the characteristics of this histogram?*
- *What would you label the y-axis of this histogram? Why?*
- *What do our histograms tell us about the likelihood of the use of different weight amounts?*

After some time, have groups share their findings and revise their own findings using other groups' reasonings.

Adapted from YouCubed

Note: Lessons from the YouCubed.org website may require setting up a free account and logging in to access the whole lesson.

In Example 2, students are:

- given a cognitively demanding task with several accessible entry points to reason about the distribution of weight usage (LP 1).
- working collaboratively in groups to complete the task, allowing for social interaction with peers. Students are also encouraged to revise their own thinking during the class discussion (LP 2).
- given a task that centers reasoning as students work to make sense of and explain the shape and feature characteristics of the distribution of weight usage. Students build understanding as they work to develop the histogram of this distribution and justify their reasoning (LP 3).
- using a human experience, such as going to the gym, to guide their thinking about what mathematics could be used to describe. They use mathematics to help describe a human behavior, such as the likelihood of lifting a certain amount of weight at the gym (LP 5).

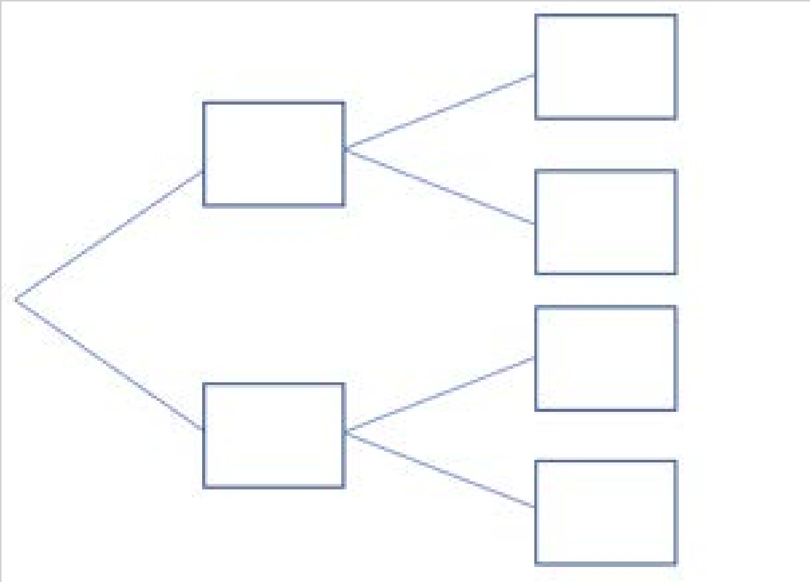
Example 3

Students are given the following prompt:

Today you will play a game in pairs with the rules as follows: Start with five cards total, two aces and three kings. The player chooses their first card and records the results, and then chooses their second card from the remaining cards and records the result. The player wins if they get a pair of aces or a pair of kings.

Choose one person to play first and play the game 10 times. Record the first card, second card, and if that game resulted in a win or loss for all 10 games. Calculate the proportion of times won. Then, the next person will play and repeat the process.

Using the tree diagram, list out all possible scenarios for each of the two rounds of this game in the boxes. In each box, also include that outcome's probability.



Discuss in your pairs:

- *Does the probability of drawing an ace/king change for the second draw? Why/why not?*
- *Which outcomes show a winning scenario?*
- *What else do you notice or wonder about the tree diagram?*
- *What can this diagram be used for?*

Write the proportion of times won in your pair on sticky dots and add the two sticky dots to the chart paper at the front of the room on their appropriate places on the dotplot.

Sample Learning Experience

Demonstrate the game with a couple of nearby students to illustrate how the game works. Be sure to show the students all five cards before the first draw, and all four cards before the second draw.

Ask students to discuss in pairs, then as a class:

- *What do you notice or wonder about the dotplot?*
- *What is the probability of winning this game? How do we know?*
- *What connection can we make between the tree diagram and the dotplot?*
- *Based on the dotplot, is it reasonable for someone to win this game eight times? nine times? 10 times? Why or why not?*
- *If we each played this game 20 times instead, would that change the tree diagram or dotplot? Why or why not?*

As students discuss, be sure to monitor, select, and sequence students who notice: that they can multiply down the branches of the tree diagram; the mean of the dotplot is about 0.4; the sum of the ace, ace and king, and king probabilities is 0.4; the probability of winning nine or 10 times is very unlikely because the dotplot shows little to no values there; the dotplot appears symmetric and bell-shaped; and the spread of the dotplot would decrease if the game is played 20 times instead of 10.

Adapted from [StatsMedic]

In Example 3, students are:

- given a cognitively demanding task. They are asked to consider the probabilities within a relatively simple game (LP 1).
- working collaboratively in pairs and sharing their findings with the class. Students are encouraged to ask questions to their peers to understand each other's thinking process (LP 2).
- invited to give a context to dependent probabilities to support their ability to communicate their conceptual understanding of conditional probabilities instead of focusing on the procedural steps for solving. Students also work to reason about the probabilities in terms of a tree diagram and a sampling distribution, making way for a deeper understanding of Bayesian reasoning (LP 3).
- given the opportunity to develop their mathematical identity, since their ideas and reasoning are the foundation of the discussion (LP 4).

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M214 Statistical Error and Predictive Model Validation

Badge Catalog Description

How accurate is a model for predicting your shoe size based on your height? What tools can you use to assess how far off these predictions are? What are the pros and cons of having less error in a predictive model? It is very rare to create a perfect predictive model, so how do you know whether to trust the models you create?

In M214 Predictive Model Validation, you will explain and quantify variation and its sources while mastering multiple methods of model validation, uncertainty, and comparison to inform decisions. You will use cross-validation methods to evaluate or compare models. Using technology, you will generate residual plots to assess regression models. You will learn how to read ANOVA tables and how to use an F-test for model evaluation or comparison. You will learn about the balance between bias and variance, and that a model with low variance does not necessarily equate to the best possible model. Your ultimate goal will be to answer the question “What might happen in the future?” based on prior knowledge—e.g., your data. You may use this information to diagnose a problem and draft its solution. Statistical error and predictive model validation are useful for careers in a variety of fields, like actuarial science, engineering, public health, and environmental sciences.

Suggested prerequisites for this badge: M212 Predictive Modeling; M213 Bayesian Reasoning and Probability Theory.

This badge is suggested as a prerequisite for: M215 Inference and Making Conclusions.

The M214 Predictive Model Validation badge puts variability at the center of how students assess their predictive models. With opportunities to develop, justify, and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving skills as they engage with various methods of assessing the error within their models. In M214, students assess the validity of predictive models, such as a model to predict Algebra 1 Test Scores using Percent of School Days Attended as a predictor.

Students should notice that this relationship is not perfect; sometimes students who always attend school do well on their exams, and sometimes these students do not do well. When fitting a linear predictive model to this data, they may notice that a nonlinear relationship may be a better fit; they can manipulate variables in hopes that their model is improved. In doing so, they can assess their model’s improvement by comparing Pearson’s r values and the patterns in their residual plots. As students assess and interpret error within models, they deepen their understanding of what variability means and its effect on their predictions, which leads to more fluid iterations of testing models in order to arrive at a predictive model that minimizes the error in the given data set.

According to *Catalyzing Change in High School Mathematics*, "The process of fitting and interpreting models for possible associations between ... variables requires a high level of reasoning that involves insight, good judgment, and a careful look at a variety of options consistent with the context and with the questions posed in the investigation" (National Council of the Teachers of Mathematics, 2017, pp. 78-79). By assessing and iterating through different predictive models for the same data set, students gain insights into the mathematics that confirms the validity of these models.

As students engage in M214 Predictive Model Validation, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M214 badge.

M214 Content and Practice Expectations

214.a	Demonstrate understanding of the role of variability in predictive modeling.
214.b	Use Pearson's r to evaluate predictive models.
214.c	Use residuals to assess model fit and decide if changes to the predictive model or its variables are necessary.
214.d	Compare predictive models using statistical techniques in order to improve these models and decide which model makes the best predictions.
214.e	Consider the bias-variance trade-off that occurs when making a predictive model.

Learning Principles

In M214 Predictive Model Validation, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M214

Often, coursework with error and variability is focused on performing statistical measures, such as standard deviation and interquartile range, sometimes by hand. Instead, students should be given continuous access to:

- tools within spreadsheets and coding languages that allow for a sharp focus on understanding how to analyze predictive models, rather than computing error and variability by hand (LP 7).
- online tools and resources such as databases, integrated development environments, videos, and data analysis platforms to allow for model validation (LP 7).

Specifically, in M214, students should:

- decipher which validation methods are appropriate for various types of predictive models (LP 1).
- use multiple validation methods for one model to gain multiple perspectives about its validity or for two models in order to determine which model is better (LP 1).
- represent the findings from their validation methods using graphical representations and written/spoken explanations (LP 3).
- explain what next steps are needed to improve predictive models using their validation method findings as justification (LP 3).
- be able to explore contexts containing predictive models that are organized around different scientific, social, or other topics for their predictive models, then validate these models using validation methods (LP 5).
- recognize sources of error in their surrounding environment in connection to a self-chosen predictive modeling task that is meaningful to them, allowing students to have agency in their learning (LP 5).
- make their thinking visible as they collaboratively work to validate predictive models in order to answer questions of interest (LP 2).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising other students' predictive model improvements and including social interaction as part of the learning process (LP 2).

Students should also be given opportunities to understand the ways to assess the accuracy of predictive models that authentically involve real-world situations and reflect on the relevance of their findings (LP 6).

Evidence of Learning

In M214 Predictive Model Validation, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
214.a: Demonstrate understanding of the role of variability in predictive modeling.	i. use technology to calculate measures of variability commonly used in the field of Data Science, e.g., Sum of Squares, Mean Absolute Error, variance, standard deviation, etc. and interpret the meaning of these measures in the context of the data set(s) being used.
	ii. demonstrate understanding that measures of variability can be used to assess predictive model fit.
	iii. demonstrate understanding that predictive modeling is a way to explain variation of a response.
214.b: Use Pearson's r to evaluate predictive models.	i. interpret Pearson's r in context and understand its purpose in quantifying the strength of a linear relationship.
	ii. demonstrate understanding that a correlation coefficient close to -1 or 1 does not necessarily mean that a linear model is appropriate.
	iii. anticipate the value of Pearson's r between two variables by visually assessing scatterplots before calculating its value using technology.
	iv. demonstrate understanding of the effect outliers have on the value of Pearson's r.
	v. calculate the value of R^2 using technology and interpret in context.
214.c: Use residuals to assess model fit and decide if changes to the predictive model or its variables are necessary.	i. use technology to produce residual summary statistics and residual plots for data sets and use them to assess model fit.
	ii. recognize outliers, patterns, random scatter, or heteroskedasticity in residual plots or the standard deviation of residuals to make decisions about predictive model fit, e.g., transforming variables to improve fit.
214.d: Compare predictive models using statistical techniques in order to improve these models and decide which model makes the best predictions.	i. use one or more statistical techniques to check for improved predictive model fit, e.g., F-tests, sensitivity analysis, transforming variables, adding or excluding variables, ANOVA, residual plots, normalizing variables, cross validation, etc.
	ii. compare measures of variability and Pearson's r of two or more predictive models to decide which model makes

Content and Practice Expectations	Indicators Choose an artifact where you...
	the best prediction.
	iii. use an iterative process to improve a predictive model.
	iv. make predictions using the improved predictive model and interpret these predictions in context.
214.e: Consider the bias-variance trade-off that occurs when making a predictive model.	i. demonstrate understanding of bias-variance trade-off, namely that a model with high bias tends to have low variance and a model with high variance tends to have low bias.
	ii. make decisions about the complexity of a predictive model after weighing the bias-variance trade-off within the model.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M214 Statistical Error and Predictive Model Validation (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a problem like this:

Build a bivariate model using machine learning relating variables in a baseball data set. While running Train/Test/Split in Colab, you will explore how model complexity affects the predictability of the algorithm. Run the program using Batting Average as the input target variable and Wins as the output predictor variable to answer the following questions:

- *Run the cell with a complexity of 1. What function does this create?*
- *What is the difference between train data and test data?*
- *What do the points represent? What are the inputs and outputs of this model?*
- *Which model do you feel would be most predictable?*
- *Does lower testing error or lower training error indicate a more predictable model?*
- *How would you use this model to make a prediction?*
- *What does the ERR value mean?*
- *How would you describe the complexity of this model?*
- *Does increase in complexity improve the model? What complexity value do you feel is the best choice?*

Sample Learning Experience

Students can choose different variables to plot against each other and vary the complexity level to see how it affects trends and error in train and test data.

Classroom discussion points:

- *Which variables are you interested in? When you create bivariate models, are there any surprises in the given context? (ex: players with fewer home runs may have a greater salary—are there other factors that impact salary?)*
- *What does complexity value represent? When you increase the complexity, how does this affect the trend line? Does this increase predictability?*

Adapted from YouCubed

Note: Lessons from the YouCubed.org website may require setting up a free account and logging in to access the whole lesson.

In this example, students are:

- given a cognitively demanding task. It requires that students model various individual baseball players' traits as they estimate the number of wins a team can produce. Students are also asked to consider improvements to their model. They further engage in thinking about why models change as the parameters and assumptions change (LP 1).
- encouraged to have agency in which variables they use, as they choose, reflect on, and refine their approaches. We can further increase students' agency in learning by offering a more open task (LP 4).
- applying mathematics in a relevant, interesting field. Students will notice how various traits of individual baseball players lead to considerations of baseball teams' draft picks as they aim to win the World Series (LP 6).
- able to use technology as a tool, as they employ Colab to perform their analysis (LP 7).

Example 2

Students are given a problem like this:

Here is data of iPhone sales during the opening weekends:

iPhone	Year	Units Sold (millions)
Original	2007	0.5
3G	2008	1
3Gs	2009	1
4	2010	1.7
4S	2011	4
5	2012	5
5C, 5S	2013	9
6, 6 Plus	2014	10
6S, 6S Plus	2015	13

Work with a partner to complete the following questions. Be prepared to share your findings with another group:

- *Use technology to create a scatterplot of the data with year as the explanatory variable and units sold as the response.*
- *Describe the form of the relationship.*
- *Use the technology to find the least squares regression line, and include its graph on your scatterplot.*
- *Use the least squares regression line to calculate the residual for 2007. Interpret the residual.*
- *Now graph the residuals using technology.*
- *For which points was the actual greater than the predicted? For which was it less than predicted? Identify these on your graph.*
- *Is the regression line a good fit for the data? Why or why not? Explain using the residual plot.*
- *Use various transformations of Year and/or Units Sold to check if there is a better model that*

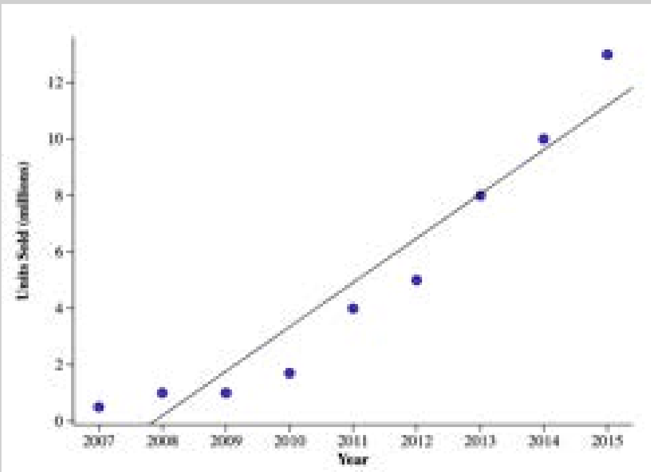
explains the relationship between Year and Units Sold.

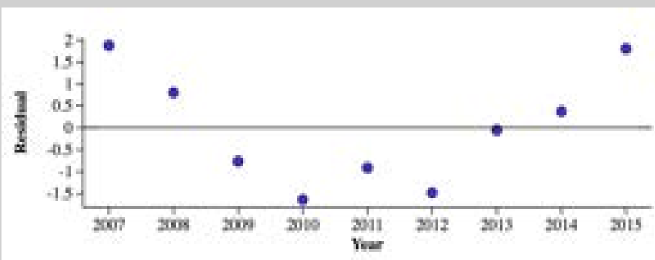
Sample Learning Experience

Before beginning the activity, pose the following question to students: "How does the computer or calculator find the line of 'best' fit? What makes it the line of best fit?" Using this Desmos eTool, let students try to move the line into the "best" spot (be sure you hide the line of best fit to start). Help them discover that the line of "best" fit is the one that minimizes the sum of the squared residuals (the least squares regression line). This can also be done nicely using the statistical software Fathom. Now students are ready for the activity!

When asked if a linear model is appropriate, students will sometimes use only the correlation value, r, to justify linearity. However, a strong correlation value doesn't mean an association is linear. An association can be clearly nonlinear and still have a correlation close to ±1. Only a residual plot can adequately address whether a line is an appropriate model for the data by showing the pattern of deviations from the line. For example, graphing the function $y = x^2$ for the integers 1 to 10 yields a correlation of r = 0.97, but the residual plot shows an obvious pattern.

Students will produce the following graphs in pairs:





Have students answer the questions, then share their findings with another pair. After some time, discuss findings as a class.

Classroom discussion points:

- *What do you notice about your two graphs? What do you wonder?*
- *Is there a type of model that would fit this data better than a line?*
- *What modifications can we make to our LSRL model to better fit this data?*

- *What similarities and differences do these graphs have?*
- *What does the residual plot tell us about the relationship between Year and Units Sold?*
- *What types of two-variable relationships appear curved?*
- *How will we know when a better relationship is accomplished?*
- *What is the value of Pearson's r in your original LSRL? Does this value fully explain the strength of the linear relationship? Why or why not?*

Adapted from StatsMedic

In this example, students are:

- working in pairs to answer questions and given opportunities to share their findings with another pair. Then, students will share their findings with the class during a whole-class discussion (LP 2).
- reasoning through questions posed to them to build conceptual understanding of why a linear relationship does not best fit a nonlinear set of data and why Pearson's r is not a fully comprehensive tool to describe linearity when they notice that the residual plot has a clear pattern (LP 3).
- given a set of relevant, real-world data to make sense of. The context of the problem is inherently meaningful due to the relevance of smartphones (LP 6).
- able to use technology as a tool, as they employ technology to produce their graphs (such as Desmos or Fathom) to perform their analysis (LP 7).

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M215 Inference and Making Conclusions

Badge Catalog Description

What is the true mean height of all students at your school? What is the true proportion of Hershey's Kisses that will land on their flat side when tossed on a table? Has the true mean carbon dioxide level of the atmosphere increased significantly over the last five years? Does a new brand of sunscreen perform significantly better on average than an old one?

In M215 Inference and Making Conclusions, you will master the logic of inference, learn a couple of inference procedures, and make decisions after weighing the risks associated with them. You will calculate and interpret P-values with technology. The confidence intervals you calculate will help inform choices about significant changes. You will learn what Type I and Type II errors are, as well as evaluate which error is worse in the context. You will use bootstrapping to improve the accuracy of your inference procedures and will make decisions based on how a study is designed and how small your P-value is. Inference and conclusion-making are useful for careers in a variety of fields, like psychology, biology, advertising, and manufacturing.

Suggested prerequisite for this badge: M214 Statistical Error and Predictive Model Validation.

The M215 Inference and Making Conclusions badge integrates probability and error as essential components of making conclusive statements from data. With opportunities to develop and revise logical arguments, students develop conceptual understanding, procedural fluency, and problem-solving as they engage with mathematical confidence intervals and statistical significance. According to *Catalyzing Change in High School Mathematics*, "Members of society encounter studies reported in the media that contain generalizations about a population or the comparison of experimental treatment groups. These studies draw inferences beyond the analysis of the data distribution by quantifying, in context, the structure that emerges in sampling distributions for a statistic such as a sample mean or sample proportion. The statistical inference process builds on data analysis and is an important component in the development of quantitative literacy. The evolution in the studying of statistical inference from exploring and describing one sample to making inferences about a population is a big leap for students" (National Council of the Teachers of Mathematics, 2017, p. 84).

As the culminating badge in the Data Science series, students utilize their knowledge of key processes and skills developed to this point to make mathematically supported conclusions about causal relationships and generalizations to populations. Ultimately, this allows students to be educated consumers of information in the world that surrounds them.

As students engage in M215 Inference and Making Conclusions, the following Content and Practice Expectations play a key role in the design of student learning experiences and student learning demonstrations for earning the M215 badge.

M215 Content and Practice Expectations

215.a	Use probability distributions to calculate p-values and interpret these p-values in context.
215.b	Use probability distributions to calculate confidence intervals in order to estimate population parameters and interpret these confidence intervals in context.
215.c	Conduct informal inference testing for quantitative and categorical data.
215.d	Use bootstrapping to estimate population parameters.
215.e	Make appropriate conclusions about study results using aspects of the study design.

Learning Principles

In M215 Inference and Making Conclusions, students will employ the following learning principles:

Engage with cognitively demanding tasks in heterogeneous settings (LP 1). Students should be given opportunities to grapple with multistep, non-routine tasks that promote mathematical rigor. These experiences should be differentiated so that all students engage in appropriate challenges, for example, through tasks with multiple entry points and solution pathways. These experiences should continue to integrate knowledge and skills developed in grades 6-8 at the level of sophistication of high school mathematics.

Engage in social activities (LP 2). Students should have opportunities to work independently and communicate with one another about mathematics by engaging in collective and collaborative learning activities. Explaining and having opportunities to revise one's thinking has excellent value for solidifying one's knowledge.

Build conceptual understanding through reasoning (LP 3). Students should be given the opportunity to reason, justify, and problem solve with critical thinking, reading, writing, speaking, and listening. By reasoning and working with multiple representations, students learn why procedures work and build conceptual understanding of key mathematical ideas.

Have agency in their learning (LP 4). Students should be able to choose tasks and learning experiences that align with their interests and aspirations. All students have rich and varied experiences and home lives. Learning mathematics should bring students' identities and interests to the fore and build on the strengths that they bring to the learning space.

View mathematics as a human endeavor across centuries (LP 5). Students should understand that mathematical ideas emanated over time from civilizations around the world and have opportunities to explore these contributions to mathematics. Students should develop an appreciation of mathematics as a human endeavor: one in which they feel a sense of belonging, where they see themselves as mathematicians, and one that offers opportunities to broaden their ideas about what mathematics is, how it is used, and who it is for.

See mathematics as relevant (LP 6). Students should engage with mathematics in ways that authentically involve real-world situations. Problem-solving contexts should allow them to see mathematics as a tool for addressing the questions that arise in everyday life, as well as the ways it can model our world and address global economic, social, and environmental challenges. Students should also engage with mathematics in ways that connect both to academic disciplines and future careers by doing mathematics used by artists, designers, engineers, and other professionals.

Employ technology as a tool for problem-solving and understanding (LP 7). Research indicates that technology is a powerful tool for learning deeper mathematics by improving calculation efficiency and enabling more sophisticated analyses. Students should learn to use technology, with emphasis put on widely used tools and software, such as calculators and spreadsheets, to make sense of models. Technology use should not be limited to supporting "doing mathematics," but should also be used as a tool for displaying and communicating results to appropriate audiences.

Points of Emphasis in M215

In M215, students should be given constant opportunities to:

- use virtual simulations through coding software to visualize and employ sampling distributions, with and without bootstrapping methods (LP 7).
- calculate p-values and confidence intervals using coding software or spreadsheets in order to focus on understanding the values' meanings in context, instead of attending so much to how these values are calculated by hand (LP 7).

Additionally, in M215, students should:

- regularly encounter real-world tasks involving the need for a parameter estimate that require them to compute and make sense of p-values or confidence intervals (LP 1).
- use open-ended tasks to reason about the context, allowing them to make decisions about what type of inference procedure and conclusion is appropriate based on the given context (LP 3).
- be able to choose tasks or entire courses that are organized around different scientific, social, or other topics, allowing students to have agency in their learning (LP 4).
- have opportunities to share their reasoning with partners or in groups, allowing for practice sharing, critiquing, and revising their arguments and including social interaction as part of the learning process (LP 2).

- regularly engage with tasks that do not require any computation, but instead focus on understanding and reasoning with the results of a study in order to draw a conclusion. For example, students should:
    - understand that causal conclusions can only happen when there is random assignment of treatment;
    - understand that generalizations to a population can only happen when there is random selection of a sample;
    - recognize that various significance levels have different potential consequences of making Type I and Type II errors;
    - interpret p-values, confidence intervals, and confidence levels;
    - use likelihood estimates to support or reject hypotheses;
    - check that appropriate conditions are met for certain inference procedures (LP 3).

Students should also be given opportunities to understand and reflect on the ways that inference and conclusion making authentically involves real-world situations (LP 6).

Evidence of Learning

In M215 Inference and Making Conclusions, students' evidence of learning can be demonstrated by the following:

(1) Portfolio of Evidence
AND
(2) Concepts and Skills Assessment

Portfolio of Evidence

Purpose: The purpose of this portfolio is to collect evidence to demonstrate that students have met the expectations for the badge over time.

Students will collect artifacts (one or more) to present evidence of their learning related to the badge content and practice expectations throughout the learning process. Students will submit evidence for each indicator listed in the table below.

Content and Practice Expectations	Indicators Choose an artifact where you...
215.a: Use probability distributions to calculate p-values and interpret these p-values in context.	i. produce a p-value using the appropriate probability distribution for a given scenario.
	ii. interpret a p-value in context and state appropriate assumptions.
215.b: Use probability distributions to calculate confidence intervals in order to	i. produce a margin of error for a parameter estimate and explain how to increase or decrease the margin of error.

Content and Practice Expectations	Indicators Choose an artifact where you...
estimate population parameters and interpret these confidence intervals in context.	ii. produce a confidence interval for a parameter estimate.
	iii. interpret a confidence interval in context and state appropriate assumptions.
	iv. interpret a confidence level in context.
215.c: Conduct informal inference testing for quantitative and categorical data.	i. describe a null and alternative hypothesis for a given scenario.
	ii. check if appropriate conditions are met for conducting a hypothesis test.
	iii. identify if a p-value supports the rejection of a null hypothesis.
215.d: Use bootstrapping to estimate population parameters.	i. use a sampling distribution generated by a bootstrap method to estimate a population parameter.
	ii. demonstrate understanding of the effect that sample size has on the bootstrap distribution.
	iii. demonstrate understanding of the effect that number of repeated samples has on the bootstrap distribution.
215.e: Make appropriate conclusions about study results using aspects of the study design.	i. describe Type I and Type II errors in the context of a given scenario.
	ii. choose a significance level for statistical significance or confidence level by weighing the consequences of Type I and Type II errors and use that significance level to draw a conclusion about a hypothesis.
	iii. use various significance or confidence levels to draw conclusions about a hypothesis.
	iv. given a description of a study with or without random assignment, determine whether there is evidence to infer a causal relationship.
	v. given a description of a study with or without random selection of a sample, determine whether results infer a generalization to the population.

Criteria for Success:

Conference and Provide Revision Support	Accept with Revision	Accept
The student's artifact shows evidence of an emerging understanding of the expectations of the indicator(s). After conferencing and additional instruction/learning, the student may provide a revised or different artifact as evidence of the indicator(s).	The student's artifact shows evidence of approaching a full understanding of the expectations of the indicator(s). The artifact may contain execution errors that should be corrected in revision. The student may revise the selected artifact or submit a different artifact.	The student's artifact demonstrates evidence that they have met the expectations of the indicator(s).

Annotated Examples M215 Inference and Making Conclusions (Optional)

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning experiences that can be done with students, both in and out of a traditional classroom setting.

Example 1

Students are given a prompt like this:

In this activity, you will use the applet at www.tinyurl.com/appletCI to learn what it means to say we are "95% confident" that our confidence interval captures the true proportion. Set the population proportion to 0.5, the confidence level to 95%, and the sample size to 75. Click "Sample" to choose a simple random sample and display the resulting confidence interval. The confidence interval is displayed as a horizontal line segment with a dot representing the sample proportion in the middle of the interval. The true proportion (p) is the green vertical line.

- *Did the first confidence interval capture the true proportion?*
- *Repeat this 10 times. How many of the intervals capture the true proportion?*

With a partner, explore this applet by generating many confidence intervals for different confidence levels, then again for different sample sizes. Use the "Sample 25" button to generate many confidence intervals at once.

Sample Learning Experience

In this activity, students will use the Confidence Intervals applet to better understand the interpretation of a confidence level and also to discover how changing the confidence level and sample size will affect the confidence interval.

Students will pair up to explore this applet. Monitor for students who notice a relationship between the percentage of intervals that capture the true proportion and the confidence level percentage. Also look for students that recognize the length of the confidence intervals differing for various reasons. Intentionally call on these students during class discussion.

Have each pair of students combine with another pair of students to create groups of four. In their groups, ask students to discuss their findings. Conduct a class discussion with the following guiding questions:

- *What did you notice?*
- *What do you still wonder?*

- *What did your group mates notice that you and your original partner did not notice?*
- *How does the confidence level affect the confidence intervals?*
- *How does the sample size affect the confidence intervals?*

After this discussion, ask students to estimate two confidence intervals by individually filling in the blanks to the following two sentences:

- I am 60% confident that tomorrow's temperature will be between _____ and _____ degrees Fahrenheit.
- I am 95% confident that tomorrow's temperature will be between _____ and _____ degrees Fahrenheit.

Ask students to discuss the differences in their two intervals in their groups. After some time, discuss as a class:

- *Were our two intervals the same or different? Why?*
- *What did your groups notice about the relationship between confidence level and the intervals?*
- *If you are using confidence intervals to decide what clothing you are going to wear, do you feel that one of the intervals is more useful than the other? Why or why not?*

Adapted from StatsMedic

In Example 1, students are:

- leveraging technology to examine confidence intervals and to support their conceptual understanding of the effect confidence level and sample size have on these intervals (LP 7).
- using the varied structures that give opportunities to determine an approach by working independently, sharing with a partner or class, and revising their response based on those interactions (LP 2).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces (LP 3).
- invited to give a context—temperature—to confidence intervals to support their ability to communicate their conceptual understanding of the meaning of confidence levels (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).

Example 2

Students are given a prompt like this:
Part 1: Joy Milne's Story
Take a couple of minutes to watch this video about Joy Milne. What questions surface for you as you listen to her story?

Part 2: A Simulation
Let's investigate whether Joy's result could have happened purely by chance, just by guessing. Working in pairs, you will simulate this study. One person will be the experimenter and one person will be Joy; then you will switch.

Important: the experimenter should not reveal the truth for each shirt. They should simply record whether the guess was correct or incorrect.

As the experimenter, keep track of the results:

Correct	Incorrect

Count up the number of correct decisions. Write the number on a sticker dot and bring it to the dotplot poster at the front of the room. Discuss with your partner what you notice and wonder about the class dotplot.

Part 3: Reflecting on this experience
During the class discussion, two terms were introduced: p-value and significance level.

1. *What do these things mean? Use examples from Milne's story and the class experiment to illustrate.*
2. *Why are these two ideas, p-value and significance level, important?*

Sample Learning Experience
Part 1: Joy Milne's Story
Begin class by showing this video about Joy Milne, a woman who claims she can smell Parkinson's Disease. Ask the class:

- *Why would it be important to know that someone can smell Parkinson's disease?*
- *How many correct decisions would you expect Joy to get out of 12 if she couldn't smell Parkinson's and she was just guessing, and why?*
- *Do we have some evidence that Joy can smell Parkinson's? Why?*
- *How many correct decisions out of 12 would it take to convince you that Joy really could smell Parkinson's?*

Part 2: A Simulation
After discussing these questions as a class, have students begin working on the simulation. Provide 12 cards: six that say "Parkinson's" and six that say "no Parkinson's." Students should shuffle the cards, guess which label each card has, and tally the results. Once the class dotplot is complete, ask the students to discuss the following questions in pairs, then discuss them as a class:

- *What assumption are we working under in this activity? Why is this important?* (Tell students that this is their null hypothesis.)
- *What hypothesis are we testing in this activity?* (Tell students that this is their alternative hypothesis.)
- *Based on the class simulation, what proportion of the simulations resulted in 11 or more correct identifications? Is this a particularly high or low amount? Is guessing 11 or more correct likely or unlikely?* (Tell students that this proportion is called a p-value.)
- *Based on these results, do we have convincing evidence that Joy can smell Parkinson's? Explain.* (Tell students that the standard cut-off, called a significance level, is typically about 5%. If the class's p-value is below 5%, then they may call the results of this activity statistically significant: thus, it provides convincing evidence.)

Part 3: Reflecting on this experience
Give students an opportunity to reflect on their learning and to share with others. As needed, provide time for students to revise and refine their work.

Adapted from StatsMedic and MrTysonStats

In Example 2, students are:

- given a cognitively demanding task. They are asked to explore and explain the likelihood of Joy's t-shirt results, under the assumption that she was guessing, using a scaffolded, hands-on activity (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers (LP 2).
- thinking about mathematics as a human endeavor as they recognize that the study was covered in the news and that they might have seen it before, and as they discuss the implications that mathematically-described results may have for the cure of a fatal illness (LP 5).
- applying mathematics in a relevant, interesting field. Students will notice how mathematics may be helpful in determining the results of any study (LP 6).

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Appendix A: Explanatory Tables

Table 1. Badge Numbers, Abbreviations, and Names

M100	QR	Quantitative Reasoning
M101	LE	Linear Equations: Concepts and Skills
M102	MFL	Modeling with Linear Functions and Equations
M103	MFQ	Modeling with Functions of Quadratic Type
M104	MFE	Modeling with Functions of Exponential Type
M111	MD1	Modeling with Data: One-Variable Measurement Data
M112	MD2	Modeling with Data: Two-Variable Measurement Data
M113	MP	Modeling with Probability
M151	MG	Modeling with Geometry
M152	GC	Geometry: Reasoning and Proof through Congruence
M153	GS	Geometry: Reasoning and Proof through Similarity
M154	GCG	Geometry: Coordinate Geometry
M155	RTT	Right Triangle Trigonometry
M201	FC	Function Concepts
M202	REC	Rational Exponents and Complex Numbers
M203	PR	Polynomial and Rational Expressions, Functions, and Equations
M204	EL	Exponential and Logarithmic Functions and Equations
M205	TF	Trigonometric Functions
M211	DMV	Data Management and Visualization
M212	PM	Predictive Modeling
M213	BRPT	Bayesian Reasoning and Probability Theory
M214	SEPMV	Statistical Error and Predictive Model Validation
M215	IMC	Inference and Making Conclusions

Table 2**.** Potential Alignment to a Typical Course Structure

Traditional Courses	High School Math Badges
Algebra 1	M101 Linear Equations: Concepts and Skills M111 Modeling with Data: One-Variable Measurement Data M112 Modeling with Data: Two-Variable Measurement Data M102 Modeling with Linear Functions and Equations

Traditional Courses	High School Math Badges
	M103 Modeling with Functions of Quadratic Type M104 Modeling with Functions of Exponential Type
Geometry	M151 Modeling with Geometry M152 Reasoning and Proof through Congruence M153 Reasoning and Proof through Similarity M154 Coordinate Geometry M155 Right Triangle Trigonometry
Algebra 2	M201 Function Concepts M202 Rational Exponents and Complex Numbers M203 Polynomial and Rational Expressions, Functions, and Equations M204 Exponential and Logarithmic Functions and Equations M205 Trigonometric Functions

Appendix B: Concepts and Skills Assessment Blueprints

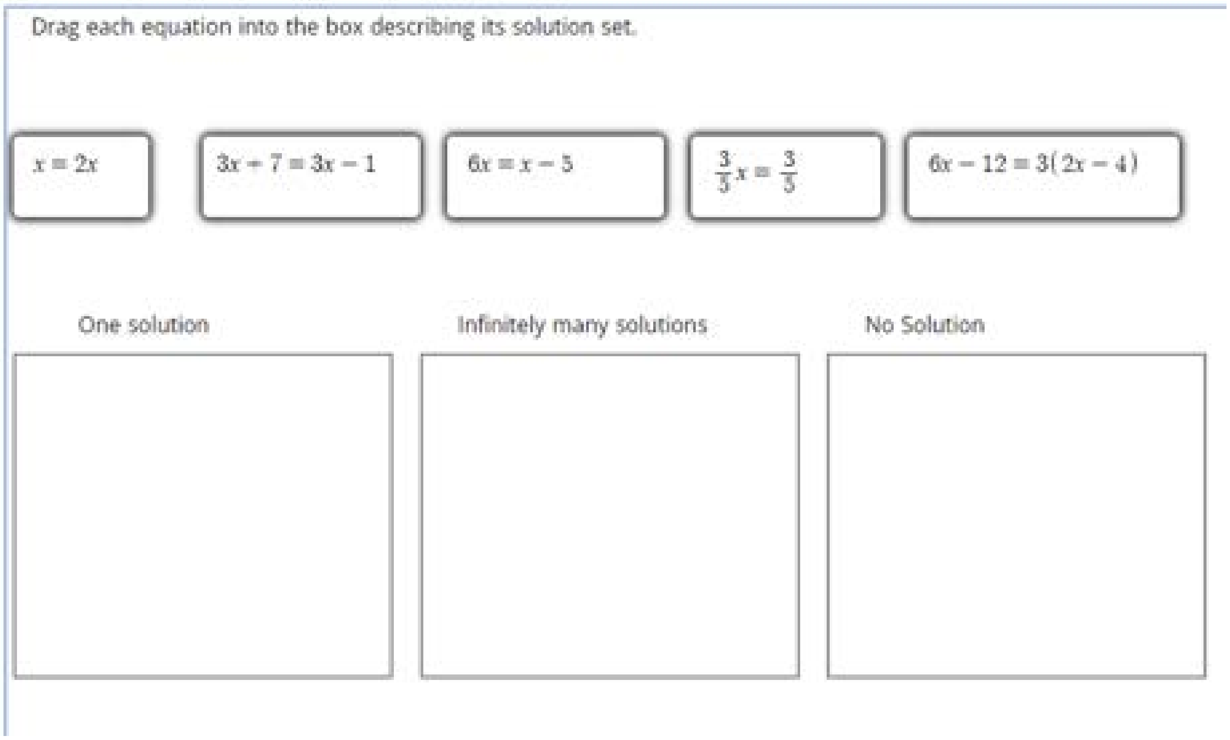
Purpose: The purpose of the Concepts and Skills assessment is to provide the opportunity for students to demonstrate evidence of their understanding of some of the concepts and skills associated with the badge.

Items: Assessment items should be rigorously tied to the content and practice expectation being assessed.

Each content and practice expectation in the suggested blueprint below includes the following elements:

- The implied subject, or the student.
- A verb stipulating the action of the student that is being assessed.
- An object which is the mathematical content of the action (e.g., The student translates between the geometric description and the equation. The objects are the geometric description and the equation).

Sample Items



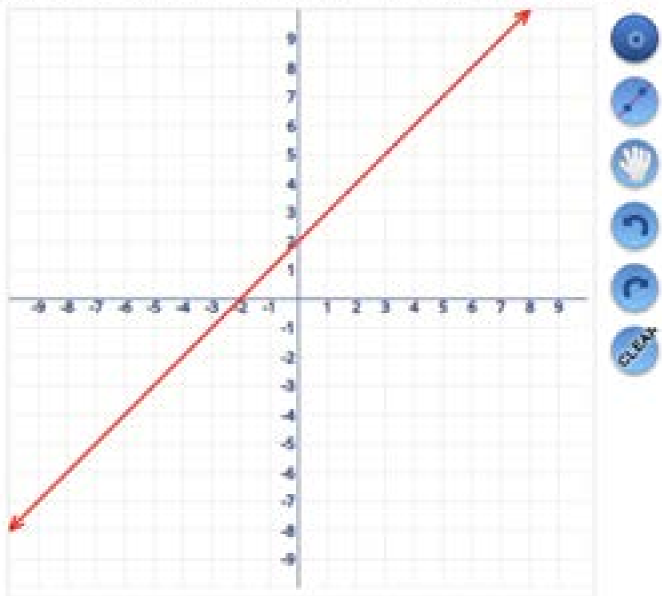
This is an example of a Concepts and Skills Assessment item aligned to M101.a.: Reason about and solve one-variable linear equations and inequalities.

Part A:

Use graphing to solve this linear equation $x + 2 = 3x - 4$. The graph of $y = x + 2$ is shown.

Use the line tool to draw another line that when graphed will show the solution for $x + 2 = 3x - 4$.

You may use the hand tool to move the line and the clear or back button to edit.



Part B:

Select an option from each drop down to complete the statement.

The [-Select an Answer-] value of the [-Select an Answer-] is the solution for the linear equation.

Sample Item 2

This is an example of a Concepts and Skills assessment item aligned to M101.d: Represent and solve linear equations and inequalities graphically.

Suggested Blueprint: M100 Quantitative Reasoning

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled by an expression, equations, tables, graphs, and data displays.	100.a: Identify quantities of interest for modeling purposes.	Application Level 2	1
	100.f: Solve real-world problems involving distances, intervals of time, liquid volume, mass, money, and other situations where operations with fractions and decimals are appropriate.	Application Level 2	2
	100.f: Solve real-world problems involving distances, intervals of time, liquid volume, mass, money, and other situations where operations with fractions and decimals are appropriate.	Application Level 1	1
	100.g: Solve real-world problems involving area and perimeter of polygons, as well as surface area and volume of prisms and pyramids.	Application Level 2	2
	100.g: Solve real-world problems involving area and perimeter of polygons, as well as surface area and volume of prisms and pyramids.	Application Level 1	1
	100.h: Solve real-world problems involving quantities in a proportional relationship, including scale drawings.	Application Level 2	2
	100.h: Solve real-world problems involving quantities in a proportional relationship, including scale drawings.	Application Level 1	1
Stimulus 2: Expression, equation, table, graph, or data display that models a theoretical scenario.	100.b: Reason with units in problems, formulas, and data displays to solve problems.	Application Level 1	2
	100.c: Interpret simple numeric and algebraic expressions that arise in applications in terms of the context.	Application Level 2	1
	100.c: Interpret simple numeric and algebraic expressions that arise in applications in terms of the context.	Application Level 1	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
	100.d: Interpret equations, tables, and graphs that arise in applications involving proportional relationships.	Application Level 2	1
	100.d: Interpret equations, tables, and graphs that arise in applications involving proportional relationships.	Application Level 1	1
	100.e: Interpret data displays such as line plots, histograms, and box plots in terms of the context.	Application Level 2	1
	100.e: Interpret data displays such as line plots, histograms, and box plots in terms of the context.	Application Level 1	1
	100.i: Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.	Application Level 2	1

Suggested Blueprint: M101 Linear Equations: Concepts and Skills

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	101.a: Reason about and solve one-variable equations and inequalities.	Conceptual Level 2	2
	101.b: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.	Procedural Level 2	2
		Application Level 2	1
	101.c: Analyze and solve linear equations and pairs of simultaneous linear equations.	Conceptual Level 2	2
		Procedural Level 2	1
	101.d: Represent and solve equations and inequalities graphically.	Conceptual Level 2	2
		Procedural Level 2	1
Stimulus 1: Two-variable equation represented graphically	101.d: Represent and solve equations and inequalities graphically.	Application Level 2	2
	101.b: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.	Procedural Level 2	1
Stimulus 2: System of two linear equations that intersect	101.c: Analyze and solve linear equations and pairs of simultaneous linear equations.	Conceptual Level 2	3
	101.d: Represent and solve equations and inequalities graphically.		

Suggested Blueprint: M102 Modeling with Linear Functions and Equations

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled by a linear function (e.g., each shirt costs $12 with a $25 one-time printing fee).	102.b: Interpret linear functions and equations that arise in applications in terms of the context.	Application Level 2	1
	102.c: Analyze linear functions using different representations.	Application Level 2	1
	102.d: Build linear functions that model relationships between two quantities.	Application Level 2	1
	102.e: Analyze and solve linear equations in two variables and pairs of simultaneous linear equations to draw conclusions.	Application Level 2	1
	102.f: Interpret expressions for linear functions in terms of the situation they model.	Application Level 2	1
	102.g: Solve linear equations and inequalities in one variable.	Application Level 1	1
	102.h: Create equations that describe linear relationships.	Application Level 2	1
	102.j: Use a linear function model to determine values of interest in a real-world problem.	Application Level 2	1
Stimulus 2: Real data set that requires students to fit a linear model. May include extraneous but relevant information.	102.b: Interpret linear functions and equations that arise in applications in terms of the context.	Application Level 2	1
	102.c: Analyze linear functions using different representations.	Application Level 2	1
	102.d: Build linear functions that model relationships between two quantities.	Application Level 2	1
	102.h: Create equations that describe linear relationships.	Application Level 2	1
Stimulus 3: Prompt/data that leads to a system of equations or inequalities in	102.d: Build linear functions that model relationships between two quantities.	Application Level 2	1
	102.e: Analyze and solve linear equations and pairs of simultaneous linear equations.	Application Level 2	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
two variables.	102.f: Interpret expressions for linear functions in terms of the situation they model.	Application Level 2	1
	102.g: Solve linear equations and inequalities in one variable.	Application Level 1	1

Suggested Blueprint: M103 Modeling with Functions of Quadratic Type

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled by a quadratic function (e.g., geometric patterns growth, gravity).	103.b: Interpret quadratic functions that arise in applications in terms of the context.	Application Level 2	2
	103.c: Analyze quadratic functions using different representations.	Application Level 2	1
	103.d: Build a quadratic function that models a relationship between two quantities.	Application Level 2	1
	103.f: Interpret expressions for quadratic functions in terms of the situation they model.	Application Level 2	1
	103.i: Use a quadratic function model to determine values of interest in a real-world problem.	Application Level 1	1
Stimulus 2: Real data set that requires students to fit a quadratic model. May include extraneous but relevant information.	103.b: Interpret quadratic functions that arise in applications in terms of the context.	Application Level 2	2
	103.c: Analyze quadratic functions using different representations.	Application Level 2	1
	103.d: Build a quadratic function that models a relationship between two quantities.	Application Level 2	1
	103.g: Summarize, represent, and interpret data on two quantitative variables for linear and quadratic model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes informal description of the fit of the curve and addresses the usefulness of the model for the particular context.	Application Level 2	1
Stimulus 3: Prompt/data that allows students to compare growth patterns modeled by linear and quadratic functions.	103.b: Interpret quadratic functions that arise in applications in context.	Application Level 2	2
	103.c: Analyze quadratic functions using different representations.	Application Level 2	1
	103.e: Construct and compare linear and quadratic models and solve problems.	Application Level 2	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
	103.g: Summarize, represent, and interpret data on two quantitative variables for linear and quadratic model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes informal description of the fit of the curve and addresses the usefulness of the model for the particular context.	Application Level 2	1

Note: More formal statistical regression analyses are part of Badge M112.

Suggested Blueprint: M104 Modeling with Functions of Exponential Type

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled by an exponential function (e.g., one person infects on average three people).	104.b: Interpret exponential functions that arise in applications in terms of the context.	Application Level 2	2
	104.c: Analyze exponential functions using different representations.	Application Level 2	1
	104.d: Build an exponential function that models a relationship between two quantities.	Application Level 2	1
	104.f: Interpret expressions for exponential functions in terms of the situation they model.	Application Level 2	1
	104.i: Use functions of exponential type to determine values of interest in a real-world problem.	Application Level 1	1
Stimulus 2: Real data set that requires students to fit an exponential model. May include extraneous but relevant information.	104.b: Interpret exponential functions that arise in applications in terms of the context.	Application Level 2	2
	104.c: Analyze exponential functions using different representations.	Application Level 2	1
	104.d: Build an exponential function that models a relationship between two quantities.	Application Level 2	1
	104.g: Summarize, represent, and interpret data on two quantitative variables for linear and exponential model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes an informal description of the fit of the curve and addresses the usefulness of the model for the particular context.	Application Level 2	1
Stimulus 3: Prompt/data that allows students to compare growth or decay modeled by linear and exponential	104.b: Interpret exponential functions that arise in applications in terms of the context.	Application Level 2	1
	104.c: Analyze exponential functions using different representations.	Application Level 2	1
	104.e: Construct and compare linear and exponential models and solve problems.	Application Level 2	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
functions.	104.g: Summarize, represent, and interpret data on two quantitative variables for linear and exponential model fits. In this badge, students are encouraged to investigate patterns of association in bivariate data, which includes an informal description of the fit of the curve and addresses the usefulness of the model for the particular context.	Application Level 2	1

Note: More formal statistical regression analyses are part of Badge M112.

Suggested Blueprint: M111 Modeling with Data: One-Variable Measurement Data

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled with one-variable measurement data.	111.b: Develop statistical questions in the course of modeling with one-variable measurement data.	Application Level 2	1
	111.e: Generate a random sample in the course of modeling with one-variable measurement data.	Application Level 1	1
Stimulus 2: One or more real data sets that students can use statistical measures and data displays to model. May include extraneous but relevant information.	111.c: Interpret differences in shape, center, and spread in the context of data sets, accounting for the possible effects of extreme data points (outliers).	Application Level 2	1
	111.d: Use measures of center and variability to describe one-variable measurement data.	Application Level 1	1
	111.f: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.	Application Level 2	1
	111.g: Compare populations by selecting and using appropriate measures of center and variability to analyze one-variable measurement data.	Application Level 1	1
	111.h: Estimate population percentages by using a normal curve to approximate a data distribution.	Application Level 2	1
Stimulus 3: One or more data displays (histogram, box plot, or line plot) based on real data that students can interpret and analyze.	111.c: Interpret differences in shape, center, and spread in the context of data sets, accounting for the possible effects of extreme data points (outliers).	Application Level 1	1
	111.d: Use measures of center and variability to describe one-variable measurement data.	Application Level 2	1
	111.f: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.	Application Level 1	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
	111.h: Estimate population percentages by using a normal curve to approximate a data distribution.	Application Level 1	1

Suggested Blueprint: M112 Modeling with Data: Two-Variable Measurement Data

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-Stimulus	112.h: Explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data.	Application Level 2	2
	112.g: Assess the fit of a function using the correlation coefficient, residuals, and other tools.	Application 2	1
Stimulus 1: Real data set that suggests a linear, quadratic, or exponential model.	112.b: Use scatter plots to represent two-variable measurement data.	Application Level 1	1
	112.c: Describe visible patterns between two data sets in a scatter plot.	Application Level 2	1
	112.e: Fit a linear, quadratic, or exponential function to two-variable measurement data.	Application Level 2	1
	112.h: Explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data.	Application Level 2	1
Stimulus 2: One or more scatter plots based on real data that students can interpret and analyze. May include a linear, quadratic, or exponential function model.	112.c: Describe visible patterns between two data sets in a scatter plot.	Application Level 1	1
	112.e: Fit a linear, quadratic, or exponential function to two-variable measurement data.	Application Level 1	1
	112.f: Use functions that have been fitted to two-variable measurement data to answer questions about the relationship being modeled.	Application Level 2	2
	112.g: Assess the fit of a function using the correlation coefficient, residuals, and other tools.	Application Level 2	1
	112.h: Explain insights gained from analyzing two-variable measurement data and limitations of curves fitted to data.	Application Level 2	2
Stimulus 3: One or more scatter plots based on real data	112.d: Informally fit a straight line for scatter plots that suggest a linear association.	Application Level 1	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
that suggests a linear relationship that students can interpret and analyze.	112.f: Use functions that have been fitted to two-variable measurement data to answer questions about the relationship being modeled.	Application Level 2	2

Suggested Blueprint: M113 Modeling with Probability

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-Stimulus Items	113.b: Understand and evaluate random processes underlying statistical experiments.	Application Level 2	4
	113.c: Use a model to determine the probability of an event.	Application Level 2	3
	113.d: Use the rules of probability to compute probabilities of compound events.	Application Level 2	2
	113.e: Understand independence and conditional probability and use them for modeling.	Application Level 2	3
	113.f: Use probability to make decisions.	Application Level 2	2

Suggested Blueprint: M151 Modeling with Geometry

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	151.c: Apply concepts of density based on area and volume in modeling situations.	Application 2	2
	151.d: Apply geometric methods to solve design problems.	Application 2	1
	151.f: Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.	Application 2	2
	151.h: Use the Pythagorean Theorem to solve right triangles in applied problems.	Application 2	1
	151.e: Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. For some examples, see http://tasks.illustrativemathematics.org/HSG-GMD.A.	Conceptual 2	2
Stimulus 1: Contextual prompt involving right triangle(s) (Illustrative Mathematics, 2016)	151.d: Apply geometric methods to solve design problems.	Application 2	1-2
	151.h: Use the Pythagorean Theorem to solve right triangles in applied problems.	Application 2	1-2

Suggested Blueprint: M152 Reasoning and Proof Through Congruence

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	152.a: Use definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.	Conceptual Level 2	1
	152.e: Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are also congruent.	Conceptual Level 2	1
	152.f: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.	Conceptual Level 2	1
	152.g: Prove theorems about lines and angles.	Conceptual Level 2	2
	152.h: Prove theorems about triangles.	Conceptual Level 2	2
	152.i: Prove theorems about parallelograms.	Conceptual Level 2	2
Stimulus 1: Geometric figure	152.b: Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using tools such as graph paper, tracing paper, or geometry software.	Conceptual Level 1	2
	152.j: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).	Conceptual Level 2	1
Stimulus 2: Two congruent geometric figures	152.c: Specify a sequence of transformations that will carry a given figure onto another.	Conceptual Level 2	1
	152.d: Given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.	Conceptual Level 2	1

Suggested Blueprint: M153 Reasoning and Proof Through Similarity

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	153.c: Verify experimentally the properties of dilations given by a center and a scale factor.	Conceptual Level 1	1
	153.e: Explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.	Conceptual Level 2	1
	153.f: Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.	Conceptual Level 2	1
	153.g: Use similarity to prove theorems about triangles.	Conceptual Level 2	3
	153.h: Use similarity criteria for triangles to solve problems and to prove relationships in geometric figures.	Application Level 2	1
		Conceptual Level 2	2
	153.a: Given a geometric figure and a dilation, draw the transformed figure using tools such as graph paper, tracing paper, or geometry software.	Conceptual Level 2	2
	153.b: Specify a sequence of transformations that will carry a given figure onto a similar figure.	Conceptual Level 1	2
	153.d: Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.	Conceptual Level 2	2

Suggested Blueprint: M154 Geometry: Coordinate Geometry

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	154.a: Translate between the geometric description and the equation for a conic section, specifically of a circle and parabola.	Conceptual Level 2	2
	154.b: Use coordinates to prove simple geometric theorems algebraically.	Conceptual Level 2	2
	154.c: Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.	Conceptual Level 2	2
		Application Level 2	2
	154.d: Find the point on a directed line segment between two given points that partitions the segment in a given ratio.	Conceptual Level 2	2
		Application Level 2	2
	154.e: Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.	Conceptual Level 2	1
		Application Level 2	2

Suggested Blueprint: M155 Right Triangle Trigonometry

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	155.a: Use similarity to explain the meaning of trigonometric ratios.	Conceptual 2	2
	155.b: Explain and use the relationship between the sine and cosine of complementary angles.	Conceptual 2	2
	155.c: Use trigonometric ratios to solve right triangles in applied problems.	Conceptual 1	2
		Conceptual 2	2
		Application 1	2
		Application 2	2

Suggested Blueprint: M201 Function Concepts

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	201.a: Understand the concept of a function and use function notation.	Application Levels 1 and 2	2
	201.b: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.	Application Levels 1 and 2	4
	201.c: Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.	Conceptual Level 2	2
		Application Levels 1 and 2	2
	201.d: Build new functions from existing functions.	Conceptual Level 2	2
		Application Level 1	1
	201.e: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).	Conceptual Level 2	2
	201.f: Build a function that models a relationship between two quantities.	Application Level 1 Conceptual Level 2	1

Suggested Blueprint: M202 Rational Exponents and Complex Numbers

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	202.a: Reason about and extend the properties of exponents to rational exponents.	Conceptual Level 2	2
		Procedural Level 2	2
	202.b: Reason about and perform operations with complex numbers.	Conceptual Level 2	2
		Procedural Level 2	2
	202.c: Analyze and use complex numbers in polynomial equations.	Procedural Level 2	2
		Conceptual Level 2	2

Suggested Blueprint: M203 Polynomial and Rational Expressions, Functions, and Equations

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	203.a: Interpret the structure of polynomial and rational expressions.	Conceptual Level 1	1
		Conceptual Level 2	1
	203.b: Reason about and perform operations on polynomials.	Procedural Level 2	3
		Conceptual Level 2	2
	203.c: Understand the relationship between the zeros and factors of polynomials.	Conceptual Level 2	3
	203.d: Use polynomial identities to solve problems.	Conceptual Level 2	1
	203.e: Rewrite rational expressions.	Procedural Level 2	1
	203.f: Interpret polynomial and rational functions that arise in applications in terms of the context.	Application Level 2	2

Suggested Blueprint: M204 Exponential and Logarithmic Functions and Equations

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Non-stimulus items	204.a: Understand the relationship between exponential and logarithmic functions.	Conceptual Level 2	2
	204.b: Graph exponential and logarithmic functions.	Conceptual Level 2	3
	204.c: Interpret logarithmic and exponential functions that arise in applications in terms of the context.	Application Level 2	4
	204.d: Use logarithms to analyze exponential models.	Application Level 2	3

Suggested Blueprint: M205 Trigonometric Functions

NOTE: This blueprint suggests stimuli for single CPEs to ensure that we can use appropriate contexts and not overwhelm students with too many different contexts.

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus	205.b: Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers.	Conceptual	2
Stimulus: Context and graph	205.e: Use trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.	Application 1	3
Stimulus: Graph, equation, data...	205.d: Analyze trigonometric functions using different representations.	Application 2	3
Non-stimulus	205.a: Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.	Conceptual 2	2
Non-stimulus	205.c: Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.	Conceptual 2	2

Suggested Blueprint: M211 Data Management and Visualization

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: One or more large data sets containing both or only quantitative and/or categorical variables that students can use statistical measures and data displays to explore. May include extraneous but relevant information.	211.b: Identify the subjects and variables in a data set.	Application 2	1
	211.d: Describe key features of raw quantitative data and categorical data.	Application 2	1
	211.e: Create graphical representations of quantitative data and categorical data.	Application 2	2
	211.f: Prepare raw data for predictive modeling by organizing, cleaning, summarizing, aggregating, and feature engineering.	Application 2	2
Stimulus 2: One or more data visualizations containing both or only quantitative and/or categorical data.	211.a: Identify features that make some graphs of quantitative and categorical data misleading.	Application 2	1
	211.c: Interpret and compare graphical representations of quantitative data and categorical data.	Application 2	2
Stimulus 3: Hypothetical scenario describing a need for data that students can use to design a process for data collection.	211.g: Understand where data comes from and how to collect primary and secondary sources of data.	Application 2	1
	211.h: Perform data practices (collecting, generating, analyzing, and disseminating data) ethically, and evaluate data practices on their use of ethics.	Application 2	1

Suggested Blueprint: M212 Predictive Modeling

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that may use a predictive model to answer a question.	212.b: Construct questions that can be answered by predictive modeling.	Application Level 2	1
	212.c: Select appropriate types of predictive models that help answer statistical questions and justify these choices.	Application Level 2	2
	212.e: Understand and implement machine learning methods and distinguish between supervised and unsupervised learning methods.	Application Level 2	1
	212.f: Improve predictive model fit by transforming variables, selecting variables, or using an alternate type of method.	Application Level 2	1
	212.g: Use predictive models to make predictions and interpret these predictions in context.	Application Level 2	2
Stimulus 2: Real data set that requires students to fit a predictive model. May include extraneous but relevant information.	212.b: Construct questions that can be answered by predictive modeling.	Application Level 2	2
	212.c: Select appropriate types of predictive models that help answer statistical questions and justify these choices.	Application Level 2	1
	212.d: Demonstrate understanding that Data = Model + Error, where the Model may contain one predictor, multiple predictors, or no predictors at all.	Application Level 2	2
	212.e: Understand and implement machine learning methods and distinguish between supervised and unsupervised learning methods.	Application Level 2	2
	212.h: Distinguish between correlation and causation.	Application Level 2	1
Stimulus 3: Prompt/data that allows students to compare multiple	212.b: Construct questions that can be answered by predictive modeling.	Application Level 2	2
	212.c: Select appropriate types of predictive	Application Level 2	1

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
types of predictive models and identify which model is best.	models that help answer statistical questions and justify these choices.		
	212.d: Demonstrate understanding that Data = Model + Error, where the Model may contain one predictor, multiple predictors, or no predictors at all.	Application Level 2	2
	212.g: Use predictive models to make predictions and interpret these predictions in context.	Application Level 2	2

Suggested Blueprint: M213 Bayesian Reasoning and Probability Theory

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Theoretical scenario that can be modeled with probability.	213.a: Understand that probability is a long run relative frequency.	Application Level 2	1
	213.b: Determine probabilities of single and multiple events using Bayesian reasoning.	Application Level 2	1
	213.c: Reason about the mathematical consequences of certain prior events.	Application Level 2	2
	213.e: Understand how sampling distributions, developed through simulation, are used to make likelihood predictions.	Application Level 2	2
Stimulus 2:	213.a: Understand that probability is a long run relative frequency.	Application Level 2	1
	213.c: Reason about the mathematical consequences of certain prior events.	Application Level 2	1
	213.d: Make likelihood predictions for discrete and continuous probability distributions.	Application Level 2	2

Suggested Blueprint: M 214 Statistical Error and Predictive Model Validation

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: One large data set containing both or only quantitative and/or categorical variables that requires a predictive model.	214.b: Use Pearson's r to evaluate predictive models.	Application Level 2	1
	214.d: Compare predictive models using statistical techniques in order to improve these models and decide which model makes the best predictions.	Application Level 2	3
	214.e: Consider the bias-variance trade-off that occurs when making a predictive model.	Application Level 2	1
Stimulus 2: Two predictive models for the same data set; one is a linear predictive model without transformed variables and one is a linear model with transformed variables.	214.a: Demonstrate understanding of the role of variability in predictive modeling.	Application Level 2	1
	214.b: Use Pearson's r to evaluate predictive models.	Application Level 2	3
	214.c: Use residuals to assess model fit and decide if changes to the predictive model or its variables are necessary.	Application Level 2	1
	214.d: Compare predictive models using statistical techniques in order to improve these models and decide which model makes the best predictions.	Application Level 2	2
Stimulus 3: Two residual plots and corresponding summary statistics data for different predictive models of the same data set.	214.a: Demonstrate understanding of the role of variability in predictive modeling.	Application Level 2	1
	214.c: Use residuals to assess model fit and decide if changes to the predictive model or its variables are necessary.	Application Level 2	1
	214.d: Compare predictive models using statistical techniques in order to improve these models and decide which model makes the best predictions.	Application Level 2	1

Suggested Blueprint: M215 Inference and Making Conclusions

Stimulus	Content and Practice Expectations	Cognitive Complexity Focus	Items
Stimulus 1: Real data set of a sample from a population that requires an estimate of the population parameter.	215.b: Use probability distributions to calculate confidence intervals in order to estimate population parameters and interpret these confidence intervals in context.	Application Level 2	3
	215.d: Use bootstrapping to estimate population parameters.	Application Level 2	2
	215.e: Make appropriate conclusions about study results using aspects of the study design.	Application Level 2	2
Stimulus 2: Real data set of a sample from a population that requires an estimate of the population parameter. May include extraneous but relevant information.	215.a: Use probability distributions to calculate p-values and interpret these p-values in context.	Application Level 2	2
	215.c: Conduct informal inference testing for quantitative and categorical data.	Application Level 2	3
	215.e: Make appropriate conclusions about study results using aspects of the study design.	Application Level 2	4

Appendix C: Performance Assessment

The performance assessment provides students with the opportunity to show evidence of their learning specific to the particulars of badges that center mathematical modeling and application, allowing them authentic ways to demonstrate their ability to apply transferable and real-world skills. For modeling badges, students will complete the full modeling cycle grounded in engaging and meaningful context(s) and report their conclusions using modes and means that are well suited to the overall purpose of the task and that allow for some agency. The following criteria are offered for designing high-quality performance assessments (Safir & Dugan, 2021):

1. Elicits evidence of skills and knowledge matter.
2. Is tight on quality criteria while open to different approaches.
3. Is authentic.
4. Offers a learning experience in and of itself.

Modeling Cycle Component[3]	Description of the Component	Cognitive Complexity Focus
Problem	In this part of the task, students are likely to engage in several of the following: • Select a topic of interest from a predetermined list or bring forth a topic of interest to their teacher. • Formulate questions to explore using the linear modeling cycle. • Co-craft a question with a peer or teacher that will align to the Content and Practice Expectations. • Select which variables to use from a table or other data set. • Find raw data that will help answer the question. • Determine the audience that they will be speaking to when answering the question.	Application Level 3
Formulate	In this part of the task, students are likely to engage in several of the following: • Identify from the data which variable(s) will be independent and dependent variable(s) based on the initial question. • Use technology—or in simple cases, by hand—to create a model using the selected variables.	

[3] Components are not necessarily a sequential list as the modeling cycle often requires iteration.

	• Define the assumptions being made with the selected variables, indicating restrictions from the context that may arise. • Decide values for the parameters of the model justified in terms of the situation being modeled and the questions being answered.	
Compute	In this part of the task, students are likely to engage in the following: • Compare the model with the raw data given to determine if the model is a good fit. • Use the model to input selected values for the independent variable to determine corresponding output values.	
Interpret	In this part of the task, students are likely to engage in the following: • Interpret the results of using the model with the data in the context of the original situation.	
Validate	In this part of the task, students are likely to engage in several of the following: • Determine if the interpretation makes sense in the context of the problem. • Identify data points that may not make sense in the original context and explain why. • Analyze the model to determine for which values of the domain it is useful and for which it is not. • Revise the model (formulate, compute, interpret, validate) if needed.	
Report	In this part of the task, students are likely to engage in several of the following: • Summarize their conclusions in a way that makes sense to the audience. • Use graphs, tables, or diagrams to help explain their conclusions. • Report the data from your model. • Summarize the process used to answer the original question(s). • Reflect on the process and indicate what might change if their assumptions were different.	

Criteria for Success:

Adapting "PROPEL" Framework from *Street Data* (Safir & Dugan, 2021).

Habits of mind are skills and habits students can build to become expert learners. Developing these is the goal of the performance assessment. These guiding questions are helpful to students, teachers, and designers to guide their thinking and choices in designing and completing the performance task.

- **Perspectives**: Does the student use mathematics to support an argument and demonstrate a sophisticated understanding of the topic?
- **Relevance**: Does the student effectively demonstrate and communicate the importance of what they are talking about to the intended audience?
- **Evidence:** Does the student provide compelling and accurate evidence in the form of facts, data, and related analyses to support their perspective about the topic of study? Are the conclusions reached based on their analysis valid?
- **Originality**: Does the student demonstrate original thinking, individual style, and creative problem-solving?
- **Logical Reasoning**: Does the student support their perspectives in a well-organized, logical, and convincing way? Does the student connect their evidence back to the original question under study to draw conclusions and, where appropriate, recognize the limitations of their model and/or identify additional questions to be asked?

Conference and Provide Revision Support	**Accept with Revision**	**Accept**
The student's product demonstrates an emerging understanding of the expectations of the modeling cycle and is beginning to capture the perspectives, relevance, evidence, originality, and logical reasoning demanded by the particulars of the assessment task. The teacher discusses entry points to expand thinking on each of the habits of mind relative to the task with the student. The student uses feedback to improve their submission.	The student's product is approaching a thorough understanding of the modeling cycle and captures some of the perspectives, relevance, evidence, originality, and logical reasoning demanded by the particulars of the assessment task. The student uses feedback to improve their submission.	The student's product shows evidence of a thorough understanding of the expectations of the modeling cycle and fully captures the perspectives, relevance, evidence, originality, and logical reasoning demanded by the particulars of the assessment task.

M102 Performance Task
The Cost of Driving Fast

Modeling with Linear Functions and Equations
Teacher Version

Design Notes

In this task, students will use linear functions to model the relationship between distance and time for several different driving speeds. This performance task is structured to guide students through the process of simultaneously engaging in each component of the modeling cycle and developing one portion of a safe-driving campaign. Students will be recording information as they go through different sections. This can either be captured in an online word processing document or on some standard lined paper if working by hand. Students will also likely want to use either graphing software or graph paper for some portions of the task. Student-teacher consultations have been built in to scaffold the process. Consider using those structured check-in points as places where select student(s) share their approach/findings with the class to support the collective learning of the class. This learning experience centers around the CCSSM modeling cycle as illustrated below:

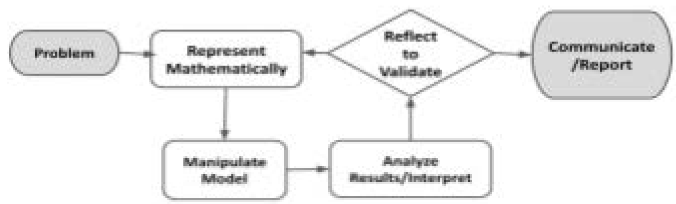


Figure 1. *Diagram of Modeling Process.* From "Progressions for the Common Core State Standards for Mathematics" [Draft manuscript], by Common Core Standards Writing Team, 2019, p. 6 (http://mathematicalmusings.org/wp-content/uploads/2022/05/modeling_02082019.pdf).

Connections to Learning Principles

Students will:

- synthesize information from several different sources to build a coherent argument in support of the safe-driving campaign (LP1);
- engage in conversations with other students to help strengthen their collective understanding of the issue (LP2);
- decide which information is relevant to include in the communication they develop (LP4);
- explore an issue that is often central to the teenage experience—safe driving (LP6); and
- have opportunities to create graphs using graphing software and create a final product that may leverage different technologies (LP7).

The following pages provide teacher versions of this task with guidance, suggestions, and comments in red and sample student responses in *blue italics*.

Student Task

The Modeling Cycle

If you want to get somewhere faster in an automobile, you simply drive faster. Right? But how much time do people really save by putting the "pedal to the metal"? In this task, you will contribute to a safe driving campaign by analyzing and reporting on the actual amount of time that people save by driving faster.

Mathematical models are used to make sense of ideas and phenomena in our world. As illustrated in the image below, the modeling cycle is a process of identifying a problem, creating models, analyzing the story told by those models, and reporting the results to the appropriate audience. In this task, you will be engaging with the mathematical modeling cycle to understand and communicate whether speeding actually saves as much time as people may believe.

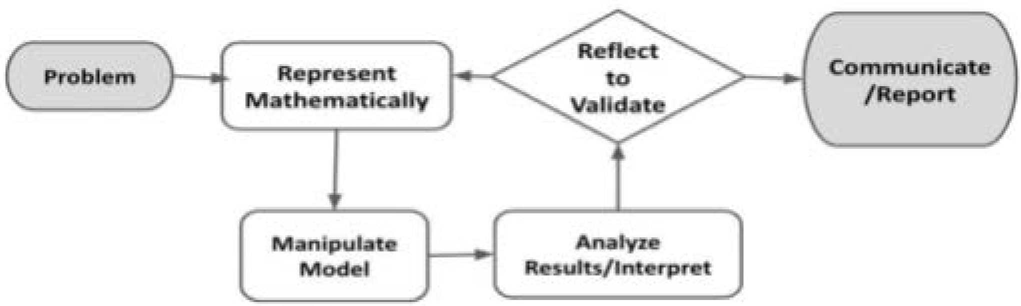


Figure 1. *Diagram of Modeling Process*. From "Progressions for the Common Core State Standards for Mathematics" [Draft manuscript], by Common Core Standards Writing Team, 2019, p. 6 (http://mathematicalmusings.org/wp-content/uploads/2022/05/modeling_02082019.pdf).

The Cost of Driving Fast

Launch

Imagine you or somebody you know gets a speeding ticket for driving faster than the posted speed limit. The speeding ticket requires a payment of $200. Was it worth driving faster than the posted speed limit, knowing that you have to pay $200?

QUESTION 1

What are some reasons why people might drive faster than the posted speed limit?

Answers will vary. Possible student responses:

- *Late for school or work.*
- *They like how it feels.*
- *To pass other cars.*
- *They don't know what the speed limit is.*

After sharing the **Launch** statement with students, consider giving students a few minutes of individual think time to generate responses.

Consider offering students various ways to share their thoughts with the class: pair-share, chart, or other spaces.

QUESTION 2

Let's think about different distances. What is a place that might be less than one mile from your current location? What is a place that might be about 20 miles from your current location? What is a place that is definitely more than 100 miles from your current location?

Answers will vary. Possible student responses:

- *[No sample answers provided as this question is very localized.]*

As you engage in this **Launch** question, feel free to tweak it to make it more meaningful. For example, if you are in a school building with students, you might ask, *What is a place that is less than one mile from our school?*

The purpose of this question is to help students think about different distances in very practical terms. They will later find out that people drive on average about 20 miles one way to get to work, so calibrating on what "20 miles" is in real terms can be very helpful to support their visualization of that later in the task.

Problem

You have been asked to create one portion of a safe-driving campaign. The primary focus of your portion of the campaign is to help drivers understand whether the potential financial cost of a speeding ticket is worth the amount of time saved by getting to a destination faster.

In the **Formulate and Represent Mathematically** section, you will examine the relationship between how fast a person is driving and how long it will take them to travel different distances. You will create one or more data displays that illuminate the relationships. The **Formulate and Represent Mathematically Checklist** can be used to ensure you have the necessary components before moving on to the remaining parts of the task.

Formulate and Represent Mathematically

QUESTION 3

How long would it take a driver traveling at 40 miles per hour (mph) to travel 10 miles? What about 100 miles?

QUESTION 4

How long would it take a driver traveling at 60 mph to travel 10 miles? What about 100 miles?

ACTION 1

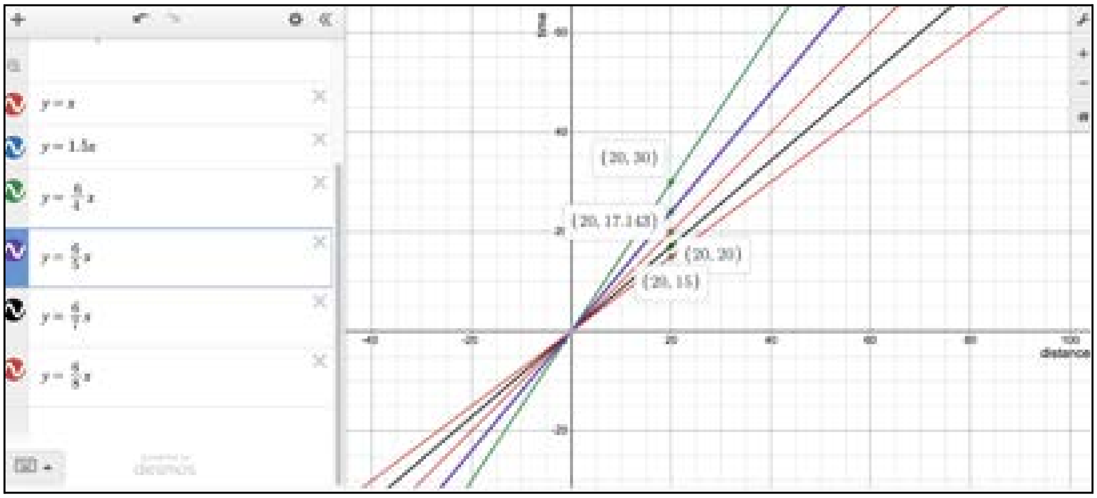
The questions presented above provide some insight into how driving at different speeds affects the amount of time it takes to get to a destination.

Create a set of graphs to show the relationship between distance traveled and time for at least three different driving speeds.

Note: *You are welcome to use additional paper or digital forms to document your work. Be sure to include all of your work in your final submission.*

The graphs you create should answer the question:
How long does it take drivers traveling at different speeds to go different distances?

See below for a sample of graphs students might create:



ACTION 2

For each graph, write the equation for that graph. Clearly define any variables used or quantities represented.

See image above.

Formulate and Represent Mathematically Checklist

Requirement	Teacher Notes
At least three graphs have been created to accurately communicate the relationship between driving speed, distance traveled, and time. Graphs are on the same set of coordinate axes.	(Teachers, please add your notes here.)
Any variables used relative to the context of the problem have been clearly defined.	(Teachers, please add your notes here.)
Each graph has been associated with an accurate equation.	(Teachers, please add your notes here.)
(Teachers, please add your notes here.)	

In the **Manipulate Model**, **Interpret**, **Reflect**, and **Report** sections, you will relate driving speed, distance, and time to inform your campaign data; you will research costs associated with speeding tickets; you will consider other information pertinent to your campaign; and you will present your findings and your final campaign. The **Manipulate Model**, **Interpret**, **Reflect**, **and Report Checklist** can be used to ensure you have the necessary components before finalizing your work on this task.

Manipulate Model

ACTION 3

Use the information from the previous section. Complete the tables below to highlight specific quantities of interest that you might use to build your final campaign.

Driving speed (in mph)	**Distance** (in miles)	**Time** (in hours)	**Time** (in minutes)
40	20	*1/2*	*30*
50	20	*2/5*	*24*
60	20	*1/3*	*20*
100	20	*1/5*	*12*

Driving speed (in mph)	**Distance** (in miles)	**Time** (in hours)	**Time** (in minutes)
40	100	*2 ½*	*150*
50	100	*2*	*120*
60	100	*1 ⅔*	*100*
100	100	*1*	*60*

Interpret

ACTION 4

Research the cost of speeding tickets locally. Decide how to integrate this information with your findings so far to make a compelling argument that the amount of time people typically save by speeding is not worth the financial burden of possibly getting a speeding ticket.

In anticipation of this local-research part of the task, you may have one or more links readily available for students to use in their research, or allow them to conduct the research on their own.

Reflect

QUESTION 5

As you get ready to synthesize your information, here are a couple of other pieces of information that might help you:

- On average, people drive about 20 miles one way to work.*
- Most people who drive faster than the posted speed limit only exceed that posted speed limit by 1-20 miles per hour.

Why is this information important to know as you develop your communication?

Are there any additional data displays or other speeds that would be valuable to examine and communicate? What other factors contribute to the speed at which a person travels?

What other costs or risks are associated with speeding? Can you find and use any other data to support your argument that speeding does not pay?

*Source: https://www.zippia.com/advice/average-commute-time-statistics/

Report

QUESTION 6

Think of different ways you can share this information with others within the context of a **public service announcement**. Specifically, aim to be as *creative* as possible. **Divergent thinking** will help you here: ***What unique idea can you generate that nobody else in your class is likely to produce?*** These types of products will not only grab your audience's attention more quickly, they will be more memorable in the long run, which, with what you now know about the costs of speeding, could help save lives!

Don't let **format** limit you—you can think of **ANYTHING** from creating/parodying music lyrics, making an informational video or commercial, writing a children's book or creating a children's rhyme they can recite when playing hopscotch or jumping rope (we're talking REALLY divergent, here!), making a speech, creating an interpretive dance, painting an abstract representative picture, assembling a collage of original related photography, or anything unique and memorable you can use to communicate your message.

Note: *You are welcome to use additional paper or digital forms to document your work. Be sure to include all of your work in your final submission.*

ACTION 5

In addition to your product, you need to create an accompanying **one-pager** with at least **one data display** (table, graph, etc.), **one paragraph** explaining the data and making the argument that speeding does not save that much time, and **one paragraph** explaining your product (this will be especially essential for products that do not use words or that convey their message implicitly/interpretively).

Manipulate Model, Interpret, Reflect, and Report Checklist

Requirement	Teacher Notes
A product has been presented that is unique/creative, well-executed, memorable, and succinct.	(Teachers, please add your notes here.)
The product has been accompanied with a one-pager that includes at least one data display, one argumentative paragraph, and one product-explanatory paragraph.	(Teachers, please add your notes here.)
Information from the task and any additional research has been effectively integrated into the product and/or one-pager to convince the audience that driving over the speed limit is not worth the	(Teachers, please add your notes here.)

financial cost (and/or other risks) of a speeding ticket.	
(Teachers, please add your notes here.)	

References

Flynn, J. (2023, February 13). *15+ average commute time statistics [2023]: How long is the average American commute?* Zippia Research. Retrieved February 16, 2023, from https://www.zippia.com/advice/average-commute-time-statistics/

Version Control Log

Date	Document title and description of the changes made to the document (if any)	Pages
7/8/22	High School Math Badging System: Working Draft This working draft includes badge frameworks for 8 of 21 badges.	1-171
8/8/22	Updated prerequisites in the High School Math Badge Catalog to clarify the relationships between badges.	16-26
9/4/22	Update the Design for the Evidence of Learning and update the Evidence of Learning sections in all Badge Frameworks to require both the Portfolio and Concepts and Skills assessments for all badges.	12-13
9/6/22	Added M100 Quantitative Reasoning	26-40
9/26/22	Updated the Badging System Map to show badges related to Algebra, Geometry, and Data Science	8
9/26/22	Updated the Concepts and Skills Blueprints for Badges M101, M102, M103, M104, and M152 based on feedback from a math assessment expert. Updates were to the number of items and the levels of Cognitive Complexity focus for the items.	47, 66, 88, 109, 143
11/3/22	Updated Geometry Badges to have unique, sequential numbers, 151-154. Updated the graphic for the Modeling Process from the *Progressions of the Common Core Standards*, Modeling, K-12 (2019).	Throughout document
11/11/22	Updated the Concepts and Skills Blueprints for Badges M153 and M154 based on feedback from a math assessment expert. Updates were to the number of items and the levels of Cognitive Complexity focus for the items.	156, 174
12/16/22	**Note:** This document includes updates to the previously published Frameworks and Concepts and Skills Blueprints for M100- M154 as well as the list of new content below. ● Additional Badge Frameworks M111-M212, M214 ● Moved Concepts and Skills Assessment Blueprints to Appendix B ● Moved Performance Assessment Description to	Throughout document

	Appendix C ● Edits to Badge Catalog Description, M204 ● Edits to Learning Principles: L1, L4, L5, L7 ● Updated content in Annotated Examples for most badges	
12/30/22	Added Badge Frameworks and Suggested Blueprints for M213 and M215	287-298, 309-318, 347, 349
2/27/23	Added Badge Catalog Descriptions for M155 and M205	28, 30
3/4/23	Updated the Badge Map, add Example Performance Assessment and Sample Items for the Concepts and Skills Assessment.	12, 347, 382
3/24/23	Added Badge Frameworks and Concepts and Skills Blueprints for M155 and M205	215-225,278-288, 367, 372
3/30/23	Full Document updated for publication. Wording change in the indicator M101.a.i. Edits for clarity on the Framework Design Principles, Learning Principles, and Evidence of Learning.	13-18, 53
4/5/23	Updated indicators M111.f.ii to remove “mean absolute deviation”	117